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Backward Stochastic Evolution Equations In Infinite Dimensions

by

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0.1 Declaration

The contents of this thesis to the best of our knowledge are our original work except where stated otherwise. We also confirm that this thesis has not been submitted for a degree at any other institution.

0.2 Acknowledgement

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0.3 Introduction

This thesis is presented to develop the theory of backward stochastic differential equations (BSDEs) and backward stochastic evolution equations (BSEEs) to infinite dimensions and to study some of their applications.

BSDEs have been widely studied over the last decade. These equations are of the following form

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t)) dt - Z(t) dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases} \quad (1)$$

where $f : [0, T] \times \Omega \times L_2(H; K) \rightarrow K$ and $\xi : \Omega \rightarrow K$ are given data such that f is progressively measurable with respect to the σ -algebra generated by the Brownian motion W , $\{\mathcal{F}_t(W), 0 \leq t \leq T\}$, and ξ is an $\mathcal{F}_T(W)$ -measurable square integrable random variable. Here H and K are some finite dimensional vector spaces and $L_2(H; K)$ is the space of Hilbert-Schmidt operators from H to K .

A solution of such an equation is a pair (Y, Z) of progressively measurable processes taking values in $K \times L_2(H; K)$ such that

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s), \quad (2)$$

for all $t \in [0, T]$.

This equation was first studied by [41] and a result of existence and uniqueness was achieved provided that the mapping f satisfies a global Lipschitz condition in the variable Y and Z . The idea of the proof of existence of solutions of (1) is based on using the usual martingale representation theorem in \mathbb{R}^n ([52]) together with a fixed point theorem.

We shall be aiming in this thesis to working with such equations under some weaker conditions than those in [41]. This is explained in detail in this introduction.

Let us now just remind the reader at this stage that the history of linear BSDEs goes back to Bismut [6]. It was shown there that linear BSDEs

may arise from some stochastic control problem, as they can be actually regarded as the adjoint equation (the equation of the adjoint process) in such a problem; see also [2].

BSDEs, increasingly, have been shown to be very useful in a number of different field of mathematics, for example, in stochastic control, see e.g. [46], [49], [50] and references therein. See also the work of Oksendal in [37] in this respect. BSDEs are applicable to some financial problems, as seen from the work of El Karoui et. al [18] and the work of Duffie and Epstein in [14] and references therein. In relation to PDEs, BSDEs have been proven to be important and useful, particularly, in giving a representation of the solution of certain PDEs. This representation generalises the so-called Feynman-Kac formula, see Section 1.4. See also [47], [1], [42], [40] and references therein for another treatment of several cases in this subject.

In Chapter 1 of this thesis we will be concentrating on the infinite dimensional version of equation (1). We shall let the spaces H and K be separable Hilbert spaces and W be a \mathcal{Q} -Wiener process in H or, more generally, a cylindrical Wiener process on H .

To be able to deal with this new equation we provide a martingale representation theorem for martingales M in K , which are adapted to the $\{\mathcal{F}_t(W), 0 \leq t \leq T\}$. This representation takes the form

$$M(t) = M(0) + \int_0^t R(s) dW(s), \quad 0 \leq t \leq T,$$

for some unique R which is progressively measurable and is Hilbert-Schmidt. Moreover, on the other hand, to be able to determine the process R in some regular cases we give a Clark-Ocone formula in this setting.

A similar result for the case W being a \mathcal{Q} -Wiener process is proved; in which case R is determined such that $R \mathcal{Q}^{1/2}$ is Hilbert-Schmidt. These are the results of Theorem 1.2 and Corollary 1.4 in Section 1.2.

Now by applying these two results, the reader may find it legitimate just

to follow the fixed point theorem argument to show the existence and uniqueness of the solution to (1). Theorem 1.9 below deals with an equation that is more general than (1), namely when replacing the integral $\int_t^T Z(s) dW(s)$ in (2) by $\int_t^T g(s, Y(s), Z(s)) dW(s)$, for some given mapping g , properties of which will be given in details in Section 1.3 of this chapter.

On the other hand, we want to know if it is possible to drop the global Lipschitz condition on f and still get a solution to (1). There are partial answers to this problem. We mention here the work of Mao in [33], where a Lipschitz condition of f on y is relaxed. In the case f taking values in \mathbb{R} there is a good progress made by Kobylanski [26], see also the work of Lepeltier and San Martin in [29] and [28]. We should note that their method depends heavily on the comparison theorem between the solutions of the BSDE, which does not seem to have a corresponding notion when moving to higher dimensions. The author does not know if their method can be generalised to investigate the higher dimensional case.

However, we find it possible to relax the Lipschitz condition slightly in y to make it satisfy a kind of *monotonicity condition* (see assumption (B3) in Section 1.3). Such a condition was used by Darling and Pardoux in [12] in finite dimensions, where they showed that can be able to approximate a mapping that satisfies this condition by a sequence of Lipschitz mappings. Their method of approximation was done by using taking a “convolution” with a sequence of smooth functions which approximate the Dirac measure at 0. This method, according to our computations, fail to work obviously in infinite dimensions. We used instead *Yosida’s approximation* for our purpose of approximating the mapping f ; see Theorem 1.14 and Proposition 1.15. This is one of our main results in Chapter 1.

Let us remind the reader here that the need to weakening the Lipschitz condition imposed on the mapping f comes from the need to study more applications. For example, to investigate the classical pricing problem in dimension one, which is linear but rarely has a bounded short rate of interest.

Some discussion on this issue can be found in [16], cf. also [18]. Another and important approach is to study BSDEs on manifolds, i.e. when the space K above is replaced, for example, by a Riemannian manifold.

In Section 1.4 we give some applications to our results in Section 1.3. First we show the property of continuous dependence of the solution of a BSDE (1) on the terminal value. Precisely, we let ξ in (1) be of the form $\xi = g(X^{t,x}(T))$ for some nice continuous mapping $g : H \rightarrow K$ and $\{X^{t,x}(s), s \geq t\}$ is a diffusion in H starting at x at time $s = t$. Then we show that the corresponding solution $Y^{t,x}$ depends continuously in L^2 on x .

Using this result we are able to study the following

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), Du(t, x) \sigma(t, x)) = 0, \\ u(T, x) = g(x), \end{cases} \quad (3)$$

where \mathcal{L} is the following time dependent second-order differential operator acting on mappings $\Psi : [0, T] \times H \rightarrow K$ as follows

$$\begin{aligned} \mathcal{L}\Psi(t, x) &: H \rightarrow K, \\ < \mathcal{L}\Psi(t, x), e'_j >_K := L\Psi_j, \quad j = 1, 2, \dots, \end{aligned} \quad (4)$$

where $\Psi_j \equiv < \Psi, e'_j >_K$, $j = 1, 2, \dots$, and L is the infinitesimal generator of $\{X^{t,x}(s), t \leq s \leq T\}$, the solution of the following S.D.E. on H

$$\begin{cases} dX^{t,x}(s) = b(s, X^{t,x}(s))ds + \sigma(s, X^{t,x}(s))dW(s), & t \leq s \leq T, \\ X^{t,x}(t) = x. \end{cases}$$

The PDE (4) is related to the BSDE (1) through taking $\xi = g(X^{t,x}(T))$. In this case we are able to represent the solution of this PDE as

$$u(t, x) = Y^{t,x}(t), \quad (t, x) \in [0, T] \times H.$$

On another point of view, we shall show also that the mapping u defined by $u(t, x) := Y^{t,x}(t)$, $(t, x) \in [0, T] \times H$, is a *viscosity solution* of (4). We shall consider two individual cases. The first case is when the Wiener process W has a finite rank, while the other case is when the mapping f in (3) does

not depend on $Du \sigma$. These are the results of Theorem(s) (1.23 , 1.24) in Section 1.4.

The subject of Chapter 2 is about backward stochastic evolution equations (BSEEs). An example of which is when the drift term f in equation (1) is unbounded. A particular case is having $A y + f(t, y, z)$ instead of $f(t, y, z)$ in (1), where A is an unbounded operator on K . Thus we have

$$\begin{cases} -dY(t) = A Y(t) dt + f(t, Y(t), Z(t)) dt - Z(t) dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi. \end{cases} \quad (5)$$

Here we let W be a cylindrical Wiener process.

These equations are useful in studying, for example, stochastic Hamilton-Jacobi-Bellman equations; cf. e.g. [48].

We shall prove in Theorem 2.1 the existence and uniqueness of mild solutions of this equation. Our assumptions on the mapping f is the same as before except that the Lipschitz condition on y is replaced by the following one where f is progressively measurable mapping which satisfies the following condition: $\exists k > 0$ such that $\forall y, y' \in K$ and $\forall z, z' \in L_2(H; K)$

$$|f(t, y, z) - f(t, y', z')|^2 \leq c(|y - y'|^2) + k |z - z'|^2,$$

uniformly in $(\omega, t) \in \Omega \times [0, T]$, where c is a continuous and nondecreasing concave function from \mathbb{R}_+ to \mathbb{R}_+ such that $c(0) = 0$, $c(x) > 0$ for $x > 0$ and

$$\int_{0+}^a \frac{dx}{c(x)} = \infty,$$

for any sufficiently small $a > 0$.

Such an equation is also studied by Hu and Peng in [25] under stronger conditions than ours here.

Furthermore, in Section 2.2 we prove some regularity properties of the solution of (5).

Section 2.3 will be devoted to describing the types of solution (weak, weak mild, strong) which equation (5) may have. We shall show actually that: strong \Rightarrow weak \Leftrightarrow mild \Leftrightarrow weak mild solution.

Moreover, a strong solution of (5) is proved, in Theorem(s) (2.18, 2.19), to exist in some cases.

We close this chapter by studying quite an important generalisation of equation (5). The operator A is allowed to be measurably time dependent, i.e. $t \mapsto A(t)y$ is Borel measurable, for all $y \in K$. This new equation which we call BSEE (evolution case), with solutions called *evolution solutions*, is studied in Section 2.4. For this case, we shall discuss the types of solutions at the end of this section.

We continue in Chapter 3 studying BSEEs, but now in more generality. The underlined filtration in Chapter 1 and Chapter 2, as introduced earlier, is the Wiener filtration. The solution (Y, Z) of BSDEs (resp. BSEEs) are adapted to this filtration. In Chapter 3 we are given an arbitrary right continuous and complete filtration, $\{\mathcal{F}_t, 0 \leq t \leq T\}$. It is useful for applications, e.g. [7], to work with arbitrary filtrations.

Chapter 3 is divide into three sections. In Section 3.1 we present a basic introduction on stochastic integration with respect to martingales in Hilbert spaces. Then we prove a kind of martingale representation theorem for \mathcal{F}_\bullet -martingales; see Theorem 3.3.

This result together with a similar one in [38] (Theorem 3.2) will be applied in Section 3.2 to study the following two types of equations.

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t)) dt - Z(t) dM(t) - dN(t), & 0 \leq t \leq T, \\ Y(T) = \xi \end{cases} \quad (6)$$

and

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t)) dt - Z(t) dW(t) - dN(t), & 0 \leq t \leq T, \\ Y(T) = \xi. \end{cases} \quad (7)$$

A solution of (6) (resp. (7)) is a triple (Y, Z, N) of predictable processes that are square integrable and satisfy the integral form (6) (resp. (7)), for each $t \in [0, T]$. Here N is required to be very strongly orthogonal (V.S.O.) to M (resp. W). This notion of orthogonality is explained in Section 3.1.

The existence and uniqueness of the solutions of equations (6) and (7) are established in Theorem 3.4 and Theorem 3.6 respectively. Our results in Theorem 3.4 can be taken as a generalisation of a result in finite dimensions got by El karoui et. al [16].

In Section 3.3 we will be dealing with equations of the type

$$\begin{cases} -dY(t) = (A(t) Y(t) + f(t, Y(t), Z(t)) + g(t, Y(t))) dt \\ \quad - Z(t) dW(t) - dN(t), \quad 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases} \quad (8)$$

where W is a cylindrical Wiener process with respect to $\{\mathcal{F}_t, 0 \leq t \leq T\}$ on H . The operator $A(t, \omega)$ is a predictable linear operator on H that belongs $L(V; V')$, where V is continuously and densely embedded in H and V' is the dual space of V . We assume also that $A(t, \omega)$ satisfies the following *coercivity* condition:

$$2 [A(t, \omega) y, y] - \lambda |y|_H^2 \leq -\alpha |y|_V^2 \quad a.e. \ t \in [0, T], \quad a.s. \ \forall y \in V,$$

for some $\alpha, \lambda > 0$. An example of this operator is given in Example 3.10.

The mapping g in this equation is a predictable V' -valued mapping defined on $[0, T] \times \Omega \times H$. The main result of Section 3.3 is to show that equation (8) has a unique solution (Y, Z, N) of predictable processes taking values in $V \times L_2(H; H) \times \mathcal{M}^{2,c}(H)$ and that Y is continuous. The space $\mathcal{M}^{2,c}(H)$ here is the space of continuous martingales taking values in H . Here also we require N to be V.S.O. to W . For this N to be continuous, we made a slight technical restriction on the filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$, which guarantees that such a martingale has always a continuous version.

Chapter 1

Introductory Results on Backward Stochastic Differential Equations in Infinite Dimensions

1.1 Preliminaries

From now on, our Hilbert spaces are supposed to be separable.

Definition 1.1 *Let H be a separable Hilbert space. Consider a symmetric positive operator $Q: H \rightarrow H$, with $\text{tr } Q < +\infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $\{W(t) : t \geq 0\}$ be an H -valued stochastic process. We say that $W(\cdot)$ is a Q -Wiener process if it satisfies the following:*

- (i) $W(0) = 0$ a.s.,
- (ii) W has continuous sample paths,
- (iii) W has independent increments, i.e.

$$\begin{aligned} & \mathbb{P} [W(t_2) - W(t_1) \in \Gamma_1 \dots, W(t_{n+1}) - W(t_n) \in \Gamma_n] \\ &= \prod_{i=1}^n \mathbb{P} [W(t_{i+1}) - W(t_i) \in \Gamma_i], \end{aligned}$$

for all $0 \leq t_1 < t_2 < \dots < t_{n+1} < \infty$ and $n \geq 1$, where $\Gamma_i \in \mathcal{B}(H)$ for all i ,

and

(iv) $W(t) - W(s)$ is a Gaussian random variable in H with mean 0 and variance $(t - s) \mathcal{Q}$.

By Gaussian here we mean that, for any arbitrary $h \in H$ the inner product $\langle W(t) - W(s), h \rangle_H$ is a Gaussian random variable in \mathbb{R} .

Let us consider the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ of subsets of Ω , as $\mathcal{F}_t = \sigma\{W(s), 0 \leq s \leq t\}$, all $t \geq 0$, completed by the probability measure \mathbb{P} .

Note that (iii) above is equivalent to the following one:

(iii)' $W(t) - W(s)$ is independent of \mathcal{F}_s , for all $0 \leq s < t < \infty$.

Thus (iii)' together with (i), (ii) and (iv) gives an equivalent definition of \mathcal{Q} -Wiener processes in H . In this case we call W a \mathcal{Q} -Wiener processes with respect to $\{\mathcal{F}_t\}_{t \geq 0}$.

Note that this definition fits well with the definition of Brownian motions in finite dimensions, e.g. in \mathbb{R} ; indeed since if W is a \mathcal{Q} -Wiener process in H , then $\langle W(\cdot), h \rangle_H$ is a constant times a 1-dimensional Wiener process, for all $h \in H$. This latter fact is one of the main ingredients needed to define stochastic integration with respect to Wiener processes in such a space H .

Let us now try to find an expansion for $W(t)$. There exists a complete orthonormal system $\{e_j\}_{j=1}^\infty$ in H and a bounded sequence of non-negative real numbers $\{\lambda_j\}_{j=1}^\infty$ such that

$$\mathcal{Q} e_j = \lambda_j e_j, \quad j = 1, 2, \dots \quad (1.1)$$

Thus one can expand $W(t)$ as the following:

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} w_j(t) e_j, \quad (1.2)$$

where

$$w_j(t) = \frac{1}{\sqrt{\lambda_j}} \langle W(t), e_j \rangle_H, \quad j = 1, 2, \dots,$$

are independent real valued Brownian motions; this follows easily from (1.1).

If we define that $\tilde{e}_j = \sqrt{\lambda_j} e_j$, $j = 1, 2, \dots$, then

$$W(t) = \sum_{j=1}^{\infty} w_j(t) \tilde{e}_j \quad (1.3)$$

and

$$w_j(t) = \frac{1}{\lambda_j} \langle W(t), \tilde{e}_j \rangle_H.$$

This series (1.3), fortunately, converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ since $\sum_{j=1}^{\infty} \lambda_j = \text{tr } Q < \infty$.

Next, we would like to take $\lambda_j \equiv 1$, all $j = 1, 2, \dots$. I.e. $Q \equiv I$, the identity map, in which case, the above series (1.3) becomes $\sum_{j=1}^{\infty} w_j(t) e_j$ which will not converge to a genuine process $W(t)$ in H . We call W a (standard) *cylindrical* Wiener process with respect to H ; see [11]. This $W(\cdot)$ can be well defined as a Q -Wiener process in a bigger Hilbert space U such that the inclusion mapping from H to U is Hilbert-Schmidt, cf. [10, Proposition 4.11, P. 96]. However, we will rarely work on this space U .

Let us now define the (natural) filtration for such cylindrical Wiener processes. We can make use of the formal expansion of W in (1.3) to define an alternative filtration for W , to be $\sigma\{w_j(s), 0 \leq s \leq t, j = 1, \dots, \infty\}$, $t \geq 0$. We will denote it by \mathcal{F}_t , for $t \geq 0$.

The following remarks on filtrations can be skipped with no harm at a first read.

Recall that an equivalent definition of a cylindrical Wiener process can also be made when regarding W as a mapping $[0, T] \times H^* \times \Omega \rightarrow \mathbb{R}$, $(t, l, \omega) \mapsto l \circ W(t, \omega)$ and such that $l \circ W(t)$ is a 1-dimensional Wiener process if $\|l\| = 1$. This enables us to define the following filtration $\mathcal{F}_t(W) = \sigma\{l \circ W(s), 0 \leq s \leq t, l \in H^*\}$, $t \geq 0$. It can be seen easily after taking limits that $\mathcal{F}_t = \mathcal{F}_t(W)$ agree for each t .

On the other hand, if we denote by \mathcal{F}_t^j the $\sigma\{w_j(s), s \leq t\}$, then $\mathcal{F}_t \subseteq \bigvee_{j=1}^{\infty} \mathcal{F}_t^j$. Also, since for each j , $w_j(\cdot) = \int_0^\cdot \langle e_j, dW(s) \rangle_H$, then we conclude

that $\bigvee_{j=1}^{\infty} \mathcal{F}_t^j \subseteq \mathcal{F}_t$, for each t . In particular, we have $\mathcal{F}_t = \bigvee_{j=1}^{\infty} \mathcal{F}_t^j$.

Suppose for a moment that we are having an *abstract Wiener space* (A.W.S.), $\iota : H \rightarrow E$, i.e. H is a separable Hilbert space included in a Banach space E via ι which is a continuous injective map with dense image and γ -*radonifying*, i.e. the push-forward measure $\iota_*(\gamma^H) = \gamma$ is a genuine measure on E , called the Wiener measure on E , where γ^H is the canonical (Gaussian) cylindrical set measure on H . As usual by identifying H with its dual, there is the adjoint of ι , $j \equiv \iota^* : E^* \rightarrow H$, such that $j(E^*)$ is dense in H with respect to $L^2(E, \gamma; \mathbb{R})$. Moreover, if $l \in E^*$ and $h \in H$, then $l \circ \iota(h) = \langle h, j(l) \rangle_H$.

We should also point out here that if $\iota : H \rightarrow E$ is an A.W.S. then $\mathcal{F}_t = \mathcal{F}_t(\tilde{W})$, where $\mathcal{F}_t(\tilde{W}) \equiv \sigma\{\tilde{W}(s), s \leq t\}$, where $\tilde{W} \equiv \iota(W)$ which is a genuine Wiener process taking values in E . To see this, first note that \tilde{W}_t is \mathcal{F}_t measurable, for each t , as seen from the definition of $\tilde{W}(t)$ as the sum $\tilde{W}(t) = \sum_{j=1}^{\infty} w_j(t) \iota(e_j)$. On the other hand, to see the other inclusion, note that for arbitrary j ; $e_j = \lim_{k \rightarrow \infty} j(l_k^j)$, where $\{l_k^j\}_{k \geq 1}$ is a sequence in E^* . Consider $\langle j(l_k^j), W \rangle_H : [0, T] \times \Omega \rightarrow \mathbb{R}$, $(t, \omega) \mapsto \langle j(l_k^j), W(t, \omega) \rangle_H = l_k^j(\iota(W(t, \omega))) = l_k^j(\tilde{W}(t, \omega))$. This implies that for each k , $\langle j(l_k^j), W(t) \rangle_H$ is $\mathcal{F}_t(\tilde{W})$ measurable, $\forall t$; hence $w_j(t)$ is $\mathcal{F}_t(\tilde{W})$ measurable since $w_j(t) = \int_0^t \langle e_j, dW(s) \rangle_H = \lim_{k \rightarrow \infty} l_k^j(\tilde{W})$. Since j is arbitrary the conclusion follows.

Denote by $L_2(H; K)$ the space of Hilbert-Schmidt operators from H into K defined by $L_2(H; K) = \{ \Phi \in L(H; K) \text{ s.t. } \sum_{j=1}^{\infty} \langle \Phi e_j, \Phi e_j \rangle_K < \infty \}$. This is a Hilbert space endowed with the norm $|\Phi|_{L_2(H; K)} = (\sum_{j=1}^{\infty} |\Phi e_j|_K^2)^{1/2}$ for any arbitrary o.n. base of H . For $T < \infty$ and a separable Hilbert space \tilde{H} let $L_{\mathcal{F}}^2(0, T; \tilde{H})$ be the space of all $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -progressively measurable processes \tilde{f} with values in \tilde{H} , (i.e. for all $t \in [0, T]$,

the process $\tilde{f} | [0, t] \times \Omega \rightarrow \tilde{H}$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -measurable,) such that

$$\mathbb{E} \int_0^T |\tilde{f}(s)|_{\tilde{H}}^2 ds < \infty.$$

Notice that $L_{\mathcal{F}}^2(0, T; \tilde{H})$ is a Hilbert space with norm

$$|\tilde{f}| = \mathbb{E} \int_0^T |\tilde{f}(s)|_{\tilde{H}}^2 ds.$$

We define a stochastic integral of processes $\Psi \in L_{\mathcal{F}}^2(0, T; L_2(H; K))$ by approximation as follows

$$\int_0^T \Psi(s) dW(s) := \lim_{N \rightarrow \infty} \sum_{j=1}^N \int_0^T (\Psi(s) e_j) dw_j(s), \quad (1.4)$$

where the integral in the right hand side now makes sense as a stochastic integral with respect to 1-dimensional Brownian motions. The limit in (1.4) exists \mathbb{P} -a.s. since

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^N \int_0^T \Psi(s) e_j dw_j(s) \right|_K^2 &= \sum_{j=1}^N \mathbb{E} \int_0^T |\Psi(s) e_j|_K^2 ds \\ &\rightarrow \sum_{j=1}^{\infty} \mathbb{E} \int_0^T |\Psi(s) e_j|_K^2 ds < \infty, \end{aligned} \quad (1.5)$$

as $N \rightarrow \infty$. Thereby $\int_0^T \Psi(s) dW(s)$ is well-defined and belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$. Furthermore, in fact, $\mathbb{E} \sup_{t \in [0, T]} \left| \sum_{j=1}^N \int_0^t \Psi(s) e_j dw_j(s) - \int_0^t \Psi(s) dW(s) \right|_K^2 \rightarrow 0$ as $N \rightarrow \infty$. Thus $\int_0^T \Psi(s) dW(s)$ can also be constructed as a limit in the above respect and is a square integrable martingale with values in K . Other equivalences and extensions of this definitions can be found in the literature.

On the other hand, note that almost the same definition can also be made to \mathcal{Q} -Wiener processes; see [10] for clear treatment of this subject, see also Métivier [35]. The definition of martingales in separable Hilbert spaces is almost the same as in finite dimensional spaces but then we have to understand the expectations through *Bochner* integration.

1.2 Martingale Representation Theorems in Infinite Dimensions

In this section we prove infinite dimensional versions of the usual finite dimensional martingale representation theorem. We will study the two cases when having a cylindrical and a genuine Wiener process in K . As a result of which we derive a Clark-Ocone formula in such a setting.

The main theorem of this section is the following.

Theorem 1.2 *Let $\{M(t), 0 \leq t \leq T\}$ be a square integrable martingale in K with respect to $\{\mathcal{F}_t\}_{t \geq 0}$, i.e. $\sup_{0 \leq t \leq T} \mathbb{E} |M(t)|_K^2 < \infty$. Then there is a unique stochastic process $R \in L^2_{\mathcal{F}}(0, T; L_2(H; K))$ such that, for all $0 \leq t \leq T$, we have a.s.*

$$M(t) = M(0) + \int_0^t R(s) dW(s). \quad (1.6)$$

In particular, M has a continuous modification.

Note that $M(0)$ in (1.6) equals to $\mathbb{E}(M(t))$, for all t .

We should mention here that this theorem was stated in [43, Theorem 1.1] without proof.

Before introducing the proof let us present some notation which we will need.

Note that since W is cylindrical it can be written formally, as in (1.3), as a infinite sum $\sum_{j=1}^{\infty} w_j(t) e_j$, where w_j , $j = 1, 2, \dots$, are i.i.d. Brownian motions in \mathbb{R} and $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of H . Define $W^N = \sum_{j=1}^N w_j e_j$ and let $\mathcal{F}_t^{(N)}$ be the σ -algebra of subsets of Ω , generated by $\{w_j(s) : 0 \leq s \leq t, j = 1, 2, \dots, N\}$. Then one can easily deduce from the definitions that $\mathcal{F}_t^{(N)} = \bigvee_{j=1}^N \mathcal{F}_t^j = \sigma\{W^N(s), s \leq t\}$, for all t . Hence

$$\mathcal{F}_t = \bigvee_{N=1}^{\infty} \mathcal{F}_t^{(N)}, \text{ for all } t.$$

For $n \in \mathbb{N}$ denote by $\pi_n : H \rightarrow H_n$, the orthogonal projection from the space H onto the finite dimensional space $H_n \equiv \langle e_1, \dots, e_n \rangle \cong \mathbb{R}^n$, which is generated by the first n elements of the basis $\{e_j\}_{j \geq 1}$.

Before giving the proof of Theorem 1.2 let us present now a simple lemma keeping in mind the above notation.

Lemma 1.3 *If $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, then there exists a sequence $\{F^N\}_{N=1}^\infty$ of random variables such that every F^N is $\mathcal{F}_T^{(N)}$ -measurable and $\mathbb{E} |F - F^N|_{\mathbb{R}}^2 \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. Take $F^N \equiv \mathbb{E} [F | \mathcal{F}_T^{(N)}]$. Then since $\sup_N \mathbb{E} |F^N|^2 < \infty$, $\{F^N, N = 1, 2, \dots\}$ is a uniformly integrable martingale. Thus F^N converges to F a.s. and in L^1 as $N \rightarrow \infty$. Being $\{F^N\}_{N \geq 1}$ bounded in L^2 implies that this convergence holds also in L^2 , as this can be seen from [52, Theorem 3.1, P. 86]. ■

Proof of Theorem 1.2. Let us first consider the one dimensional case, i.e. when M takes values in \mathbb{R} . Let $F \equiv M(T)$. From Lemma 1.3 we can approximate F by $F^N \in L^2(\Omega, \mathcal{F}_T^{(N)}, \mathbb{P}; \mathbb{R})$ such that $\mathbb{E} |F - F^N|_{\mathbb{R}}^2 \rightarrow 0$ as $N \rightarrow \infty$.

From [52, Proposition 3.2, P. 199] (or see [33, Lemma 2.4, P. 237]) we see that, for all $N \geq 1$, there exists a unique stochastic process $R^N \in L^2_{\mathcal{F}(N)}(0, T; L_2(H_N; \mathbb{R}))$ such that

$$F^N = \mathbb{E} F^N + \int_0^T R^N(s) dW^N(s). \quad (1.7)$$

By letting $\tilde{R}^N \triangleq R^N \circ \pi_N$, which then belongs to $L^2_{\mathcal{F}}(0, T; L_2(H; \mathbb{R}))$ we can re-write 1.7 as

$$F^N = \mathbb{E} F^N + \int_0^T \tilde{R}^N(s) dW(s). \quad (1.8)$$

We want to obtain such a representation for F . Note that from (1.8) we derive the following

$$\mathbb{E} \int_0^T |\tilde{R}^m(s) - \tilde{R}^N(s)|_{L_2(H; \mathbb{R})}^2 ds =$$

$$\mathbb{E} |F^m - F^N - \mathbb{E} F^m + \mathbb{E} F^N|_{\mathbb{R}}^2 \longrightarrow 0,$$

as $m, N \rightarrow \infty$. Therefore $\{\tilde{R}^N\}_{N=1}^{\infty}$ is a Cauchy sequence in $L^2_{\mathcal{F}}(0, T; L_2(H; \mathbb{R}))$, whence it has a limit in this space. Call it R . Finally, by passing the L^2 - limit through in (1.8) as $N \rightarrow \infty$, we get

$$F = \mathbb{E} F + \int_0^T R(s) dW(s).$$

which is the required formula. Hence

$$M(t) = \mathbb{E} [F | \mathcal{F}_t] = M(0) + \int_0^t R(s) dW(s).$$

It remains to prove such a representation when M takes values in the space K .

Assume for simplicity that $\mathbb{E} M = 0$. If $M(t) \in K$ and $\{\acute{e}_l\}_{l=1}^{\infty}$ is an arbitrary orthonormal basis of K , then $\langle M(t), \acute{e}_l \rangle_K$ is square integrable martingale in \mathbb{R} for each l . Hence

$$\begin{aligned} M(t) &= \sum_{l=1}^{\infty} \langle M(t), \acute{e}_l \rangle_K \acute{e}_l \\ &= \sum_{l=1}^{\infty} \int_0^t R_{\acute{e}_l}(s) dW(s) \acute{e}_l. \end{aligned}$$

Since

$$\begin{aligned} +\infty &> \mathbb{E} |M(t)|^2 &= \mathbb{E} \sum_{l=1}^{\infty} \int_0^t |R_{\acute{e}_l}(s)|_{L_2(H; \mathbb{R})}^2 ds \\ &= \mathbb{E} \int_0^t \left| \sum_{l=1}^{\infty} R_{\acute{e}_l}(s) \acute{e}_l \right|_{L_2(H; K)}^2 ds \\ &= \mathbb{E} \int_0^t |R(s)|_{L_2(H; K)}^2 ds, \end{aligned}$$

where we took $R(s) \triangleq \sum_{j=1}^{\infty} R_{\acute{e}_j}(s) \acute{e}_j$. This completes the proof. ■

This theorem applies also when having a \mathcal{Q} -Wiener process instead of a cylindrical one. We record this in the following corollary.

Corollary 1.4 Suppose W is a \mathcal{Q} -Wiener process evolving in H . Let $\{M(t), 0 \leq t \leq T\}$ be a square integrable martingale in K with respect to natural filtration of W , $\{\mathcal{F}_t\}_{t \geq 0}$. Then there is a unique stochastic process $R \in L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$ such that, for all $0 \leq t \leq T$, we have \mathbb{P} -a.s.

$$M(t) = M(0) + \int_0^t R(s) dW(s).$$

In particular, M has a continuous modification.

Here $G \in L^{\mathcal{Q}}_2(H; K) \Leftrightarrow G\mathcal{Q}^{1/2} \in L_2(H; K)$.

Proof. Note that W is cylindrical Wiener process on $\tilde{H} \equiv \mathcal{Q}^{1/2}(H)$, equipped with the inner product

$$\langle a, b \rangle_{\tilde{H}} := \langle \mathcal{Q}^{-1/2}a, \mathcal{Q}^{-1/2}b \rangle_H,$$

as this can be seen easily from expanding W as an infinite sum as we did earlier in Section 1. By using Theorem 1.2, the result then follows. ■

After we have shown that the martingale representation theorem holds in our infinite dimensional setting we may ask if a *Clark-Ocone* formula still holds in this setting. The answer is positive and we shall see below how we can find the process R appearing in (1.6) if the terminal value $M(T)$ is regular enough. Before going directly to that business let us present the following notions that we shall need.

Suppose that $\iota : H \rightarrow E$ is an A.W.S. with γ being the Wiener measure on E . Let $\mathcal{E} = C_0([0, T]; E)$ and $\mathcal{H} = L^{2,1}_0([0, T]; H)$. Then $\mathcal{I} : \mathcal{H} \rightarrow \mathcal{E}$ is an A.W.S., where $\mathcal{I}(h)(t) = \iota(h(t))$ if $h \in \mathcal{H}$; cf. [8]. Denote by Γ the corresponding Wiener measure on \mathcal{E} . Let J denote $I^* : \mathcal{E}^* \rightarrow \mathcal{H}$. Assume that $\{W(t), 0 \leq t \leq T\}$ is a cylindrical Wiener process on H .

Since \mathcal{E}^* is dense in \mathcal{H} , if $h \in \mathcal{H}$ then there exists a sequence $\{l_k\}_{k \geq 0}$ in \mathcal{E}^* such that $h = \lim_{k \rightarrow \infty} J(l_k)$ in \mathcal{H} . But $|J(l_k)|_{\mathcal{H}} = |l_k|_{L^2(\mathcal{E}, \Gamma; \mathbb{R})}$, as it is well-known from the construction of abstract Wiener spaces. Thus $\{l_k\}_{k \geq 0}$ converges in $L^2(\mathcal{E}, \Gamma; \mathbb{R})$. Let $W(h) := \lim_{k \rightarrow \infty} l_k$. Note that if $h = J(l)$, some

$l \in E^*$ then $W(J(l)) = l$ a.s. But $l(I(\dot{h})) = \langle j(l), \dot{h} \rangle_{\mathcal{H}}$ for all $\dot{h} \in \mathcal{H}$. Thus we obtain $W(J(l))(I(\dot{h})) = \langle J(l), \dot{h} \rangle_{\mathcal{H}}$. In particular, $W(h)$ generalises the inner product $\langle h, \cdot \rangle_{\mathcal{H}}$ on \mathcal{H} .

On the other hand, for any $h \in \mathcal{H}$, $W(h) = \int_0^T \langle \dot{h}(s), dW(s) \rangle_{\mathcal{H}}$ a.s., the Paley-Wiener integral, cf. e.g. [56, P.266]. Some of these remarks were discussed in [57], see also [36] and [55]. Other notation of $W(h)$ appeared in the literature as δh , $\langle h, - \rangle_{\mathcal{H}}$ and $I(h)$. For more information see [22].

Definition 1.5 *Let Ξ be a real separable Hilbert space. A function $F : \Omega \rightarrow \Xi$, is called an Ξ -valued cylindrical polynomial if it is of the form*

$$F = p(W(h_1), \dots, W(h_n)), \quad h_j \in \mathcal{H},$$

where $p(\underline{x}) = \sum_{i=1}^n p^i(\underline{x}) \dot{e}_i$, where p^i , $i \geq 1$ are real-valued polynomials on \mathbb{R}^n and $\{\dot{e}_i\}_{i \geq 1}$ is an o.n. base of Ξ . It is therefore a linear combination of functions $x^m \dot{e}$, $m \geq 0$, $x \in \mathbb{R}$ and $\dot{e} \in \Xi$. The totality of such polynomials will be denoted by $\mathcal{P}(\Xi)$.

It is well-known that $\mathcal{P}(K)$ is dense in $L^p(\Omega, \Gamma; K)$ for all $1 \leq p < \infty$, where $\Omega \equiv \mathcal{E}$; cf. e.g. [58].

Define the \mathcal{H} -gradient of such F by

$$\nabla_{\mathcal{H}} F = \sum_{j=1}^n \partial_j p(W(h_1), \dots, W(h_n)) \otimes \int_0^T \dot{h}_j(s) ds.$$

The presence of this tensor \otimes is to regard $\nabla_{\mathcal{H}}$ as a random variable $\Omega \rightarrow K \otimes \mathcal{H}$. This is denoted also by ∇F and is called the *Gross-Sobolev derivative* of F ; see [56] and [55] for the properties.

Define for $h \in \mathcal{H}$,

$$\nabla_h F = \sum_{j=1}^n \partial_j p(W(h_1), \dots, W(h_n)) \langle h_j, h \rangle_{\mathcal{H}}.$$

Thus, for fixed ω , $\nabla F(\omega) : \mathcal{H} \rightarrow K$ is a continuous linear operator for each ω .

Remark 1.6 For $F \in \mathcal{P}(K)$, define $DF(\omega)(u) = \frac{\partial}{\partial \lambda} F(\omega + \lambda u)|_{\lambda=0} \in K$, $\omega, u \in \Omega$. Then for $h \in \mathcal{H}$,

$$\begin{aligned} D_{\mathcal{H}}F(\omega)(h) &\equiv DF(\omega)(\iota(h)) \\ &= \sum_{j=1}^n \partial_j p(W(h_1), \dots, W(h_n)) < h_j, h >_{\mathcal{H}}. \end{aligned}$$

In particular,

$$\nabla_h F(\omega) = [\nabla_{\mathcal{H}}F(\omega), h]_{\mathcal{H}} = D_{\mathcal{H}}F(\omega)(h).$$

This “inner product”, $[\cdot, \cdot]_{\mathcal{H}}$, is a bilinear map from $(K \otimes \mathcal{H}) \times \mathcal{H}$ to K , defined by taking the inner product of the corresponding \mathcal{H} -valued of the first entry with the second entry to obtain an element of K . One could look at it as the following

$$[k \otimes \dot{h}, h]_{\mathcal{H}} = k \otimes < \dot{h}, h >_{\mathcal{H}} = k < \dot{h}, h >_{\mathcal{H}},$$

for $h, \dot{h}, k \in \mathcal{H}$. This $[\cdot, \cdot]_{\mathcal{H}}$ agrees with the inner product $< \cdot, \cdot >_{\mathcal{H}}$ in the case $K = \mathbb{R}$.

Define

$$D_t F(\omega) = (1_K \otimes \frac{d}{dt}) \{ \nabla_{\mathcal{H}} F(\omega)_t \}.$$

Hence $D_t F$ is a mapping from $\Omega \rightarrow K \otimes H$ and

$$D_t F = \sum_{j=1}^n \partial_j p(W(h_1), \dots, W(h_n)) \otimes \dot{h}_j(t).$$

Observe that $\nabla_{\mathcal{H}} : \mathcal{P}(K) \rightarrow \mathcal{P}(K \otimes \mathcal{H})$ and similarly $\nabla_{\mathcal{H}}^k : \mathcal{P}(K) \rightarrow \mathcal{P}(K \otimes \mathcal{H}^{\otimes k})$ for $k \geq 1$. This operator $\nabla_{\mathcal{H}}$ is closeable on all $L^p(\Omega, \Gamma; \mathbb{R})$ spaces, $1 \leq p < \infty$; see [56, P. 265]. Thus we can define the spaces $\mathbb{D}_{p,k}(K)$ to be the completion of the $\mathcal{P}(K)$ under the following norm, $\|\cdot\|_{p,k}$,

$$\|F\|_{p,k} = (\mathbb{E} |F|_K^p + \mathbb{E} |\nabla_{\mathcal{H}}^k F|_{K \hat{\otimes}_2 \mathcal{H}^{\otimes k}}^p)^{1/p},$$

where $\hat{\otimes}_2$ denotes the completed Hilbert-Schmidt tensor product. Thus $\nabla_{\mathcal{H}} : \mathbb{D}_{p,k}(K) \rightarrow \mathbb{D}_{p,k-1}(K \hat{\otimes}_2 \mathcal{H}^{\hat{\otimes}(k-1)})$ is well-defined as a linear operator. In particular, $F \in \mathbb{D}_{p,1}(K)$ if and only if there exists a sequence of cylindrical (smooth or polynomial) random variables $\{F_n : n \in \mathbb{N}\}$ converging to F in $L^p(\Omega, \Gamma; K)$ such that $\nabla_{\mathcal{H}} F_n$ is Cauchy in $L^p(\Omega, \Gamma; K \hat{\otimes}_2 \mathcal{H})$; from which $\nabla_{\mathcal{H}} F = \lim_{n \rightarrow \infty} \nabla_{\mathcal{H}} F_n$.

We conclude easily from the definition of D_t that it is also well-defined as linear map that takes every $F \in \mathbb{D}_{p,k}(K)$ to $\mathbb{D}_{p,k-1}(K \hat{\otimes}_2 H^{\hat{\otimes}(k-1)})$. So $\nabla_{\mathcal{H}}^k$ and $D_{t_1, \dots, t_k}^k \equiv D_{t_1} D_{t_2} \dots D_{t_k}$ make sense on their relevant spaces. Notice that $\|F\|_{2,1} = \mathbb{E} |F|_K^2 + \mathbb{E} \int_0^T |D_t F|_{K \hat{\otimes}_2 H}^2 dt$.

The inner product $[\cdot, \cdot]_{\mathcal{H}}$, defined earlier, can easily be extended by linearity to a continuous bilinear map : $(K \hat{\otimes}_2 \mathcal{H}) \times \mathcal{H} \rightarrow K$. We will denote it also by $[\cdot, \cdot]_{\mathcal{H}}$. This definition of $[\cdot, \cdot]_{\mathcal{H}}$ also can be made, similarly, with respect to any arbitrary separable Hilbert space, e.g. the space H .

Let us try to make use of the above definitions in the following example when dealing with a classical Wiener space.

Example 1.7 Let K and H be \mathbb{R} . Consider $F = f(W(t_1), \dots, W(t_n))$, $f \in C^\infty(\mathbb{R}^n)$. For $h \in \mathcal{H}$,

$$\mathbb{E} \langle \nabla_{\mathcal{H}} F, h \rangle_{\mathcal{H}} = \mathbb{E} \nabla_h F = \mathbb{E} \left[F \cdot \int_0^T \dot{h}(s) dW(s) \right].$$

by using the Cameron-Martin theorem. Here we have identified the elements of $\mathcal{H} \hat{\otimes}_2 \mathbb{R}$ with the corresponding ones in \mathcal{H} .

Note that $W(t_i)$ can be written as $W(h_i)$, where $h_i = \int_0^{t_i} 1_{[0,t_i]}(s) ds$. From the definitions above we then conclude immediately the following values

$$\begin{aligned} \nabla_{\mathcal{H}} F &= \sum_{j=1}^n \partial_{t_j} f(W(t_1), \dots, W(t_n)) \cdot \int_0^{t_j} 1_{[0,t_j]}(s) ds, \\ D_t F &= \sum_{j=1}^n \partial_{t_j} f(W(t_1), \dots, W(t_n)) \cdot 1_{[0,t_j]}(t), \end{aligned}$$

$$\nabla_h F = \sum_{j=1}^n \partial_i f(W(t_1), \dots, W(t_n)) \int_0^T \langle 1_{[0, t_i]}(s), \dot{h}(s) \rangle_{\mathbb{R}} ds.$$

Finally, $D_t W(s, \omega) = 1_{[0, s]}(t)$. In particular, $D_t W(s, \omega) = 0$ if $t > s$. Similar calculations can be found in [36] and [55].

We are now ready to state the Clark-Ocone theorem.

Theorem 1.8 *If $F \in \mathbb{D}_{2,1}(K)$ then*

$$F = \mathbb{E} F + \int_0^T [\mathbb{E}\{D_t F | \mathcal{F}_t\}, dW(t)]_H. \quad (1.9)$$

Proof. Since $\mathcal{P}(K)$ is dense in $\mathbb{D}_{2,1}(K)$, it is sufficient to prove the theorem for elements of $\mathcal{P}(K)$. Suppose that $F \in \mathcal{P}(K)$. Let $0 \leq t_j < t_{j+1} \leq T$ and α_j be bounded H -valued and \mathcal{F}_{t_j} -measurable. Then $k(t) := (t \wedge t_{j+1} - t \wedge t_j) \alpha_j$ is a bounded, \mathcal{F}_t -adapted process, with paths taking values in \mathcal{H} .

By Cameron-Martin theorem

$$\begin{aligned} & \mathbb{E} [F(\omega + \tau \mathcal{I}(k(\cdot))) \cdot \exp(-\tau \int_0^T \langle \dot{k}(s), dW(s) \rangle_H - \frac{\tau^2}{2} \int_0^T |\dot{k}(s)|_H^2 ds)] \\ &= \mathbb{E} [F] \end{aligned} \quad (1.10)$$

Differentiating (1.10) for τ at $\tau = 0$ yields the following

$$\mathbb{E} [\nabla_{\mathcal{H}} F, k]_{\mathcal{H}} = \mathbb{E} [F \cdot \int_0^T \langle \dot{k}(s), dW(s) \rangle_H]. \quad (1.11)$$

By linearity (1.11) holds for any bounded elementary H -valued process, k , adapted to $\{\mathcal{F}_t, 0 \leq t \leq T\}$.

Assume that $\mathbb{E} F = 0$ for simplicity, or consider $F - \mathbb{E} F$.

Let g be a bounded elementary process with values in H , adapted to $\{\mathcal{F}_t, t \geq 0\}$. Take $c \in \mathbb{R}$ and let

$$G = c + \int_0^T \langle g(s), dW(s) \rangle_H.$$

Thus $G \in L^2(\Omega, \mathcal{F}_T, \Gamma; \mathbb{R})$. By Theorem 1.2 such G are dense in $L^2(\Omega, \mathcal{F}_T, \Gamma; \mathbb{R})$. We can observe immediately, as done for (1.11), that

$$\begin{aligned}
\mathbb{E} [F \cdot G] &= \mathbb{E} [\nabla_{\mathcal{H}} F, \int_0^T g(s) ds]_{\kappa} \\
&= \mathbb{E} \int_0^T [D_s F, g(s)]_H ds \\
&= \mathbb{E} \int_0^T [\mathbb{E}\{D_s F | \mathcal{F}_s\}, g(s)]_H ds \\
&= \mathbb{E} \int_0^T [\mathbb{E}\{D_s F | \mathcal{F}_s\}, dW(s)]_H \cdot \int_0^T [g(s), dW(s)]_H \\
&= \mathbb{E} [\int_0^T [\mathbb{E}\{D_s F | \mathcal{F}_s\}, dW(s)]_H \cdot G].
\end{aligned}$$

The proof is complete. ■

1.3 Backward Stochastic Differential Equations

In this section we shall demonstrate how we can apply Theorem 1.2 or in particular Corollary 1.4 to establish some results regarding existence and uniqueness of solutions to BSDEs of type (1.16) and (1.12) (see below), first when imposing a global Lipschitz condition on the coefficients f and g . Secondly, we set a monotonicity condition on the drift f , with respect to the first variable y . We shall explain below that this latter condition is weaker than f being Lipschitz in y . These parameters appear in the following backward stochastic differential equation, BSDE in short, as follows

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t)) dt - g(t, Y(t), Z(t)) dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases} \quad (1.12)$$

where W is a \mathcal{Q} -Wiener process taking values in H . This equation is read usually in its integral form

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T g(s, Y(s), Z(s)) dW(s), \quad (1.13)$$

$0 \leq t \leq T$.

Considering the natural filtration for this W now, also denoted by $\{\mathcal{F}_t, 0 \leq t \leq T\}$, we assume that f is a mapping from $[0, T] \times \Omega \times K \times L_2^{\mathcal{Q}}(H; K)$ to H that is $\mathcal{P} \otimes \mathcal{B}(K) \otimes \mathcal{B}(L_2^{\mathcal{Q}}(H; K)) / \mathcal{B}(K)$ -measurable, where \mathcal{P} is the σ -algebra of \mathcal{F}_t -progressively measurable subsets of $[0, T] \times \Omega$. Also g is a mapping from $[0, T] \times \Omega \times K \times L_2^{\mathcal{Q}}(H; K)$ to K , and is assumed to be $\mathcal{P} \otimes \mathcal{B}(K) \otimes \mathcal{B}(L_2^{\mathcal{Q}}(H; K)) / \mathcal{B}(K)$ -measurable.

Let us now put some conditions on these mappings f, g and ξ .

- (A1) The two mappings $f(\cdot, 0, 0), g(\cdot, 0, 0) \in L_{\mathcal{F}}^2(0, T; K), L_{\mathcal{F}}^2(0, T; L_2^{\mathcal{Q}}(H; K))$, respectively and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$
- (A2) $\exists k > 0$ such that $\forall y, y' \in K$ and $\forall z, z' \in L_2^{\mathcal{Q}}(H; K)$

$$|f(t, y, z) - f(t, y', z')|_K^2 \leq k (|y - y'|_K^2 + |z - z'|_{L_2^{\mathcal{Q}}(H; K)}^2),$$

and

$$|g(t, y, z) - g(t, y', z')|_{L_2^{\mathcal{Q}}(H; K)}^2 \leq k (|y - y'|_K^2 + |z - z'|_{L_2^{\mathcal{Q}}(H; K)}^2),$$

$(t, \omega) dt \times \mathbb{P} a.e.$

- (A3) $\exists \alpha > 0$ such that

$$|g(t, y, z) - g(t, y, z')|_{L_2^{\mathcal{Q}}(H; K)}^2 \geq \alpha |z - z'|_{L_2^{\mathcal{Q}}(H; K)}^2,$$

$\forall z, z' \in L_2^{\mathcal{Q}}(H; K)$ and $\forall y \in K, (t, \omega) a.e.$

- (A4) For each (t, ω, y) the mapping $z \mapsto g(t, y, z)$ is surjective on $L_2^{\mathcal{Q}}(H; K)$.

From (A3) and (A4) we see that the mapping $z \mapsto g(t, y, z)$ is bijection on $L_2^{\mathcal{Q}}(H; K)$. Note that in finite dimensions, e.g. $\mathbb{R}^n, n \geq 1$ a mapping satisfying (A2) and (A3) should be a bijective mapping; see an argument on this in [41]. However, this is not the case, in general, in infinite dimensions

since, for example, there exists a diffeomorphism between a separable Banach space E and $E - 0$ of bounded support, cf. [15] and reference therein.

A *strong* solution of (1.12) is a pair (Y, Z) in $L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$, such that (1.13) holds.

The following theorem is an infinite dimensional version of [41, Theorem 4.1].

Theorem 1.9 *Under (A1)–(A4) the BSDE (1.12) attains a unique strong solution $Y(\cdot), Z(\cdot)$ in $L^2_{\mathcal{F}}(0, T; K), L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$, respectively.*

Remark 1.10 *It is worth noting that Theorem 1.9 still holds also when replacing W by cylindrical Wiener process, in which case the solution (Y, Z) lies in $L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$, where $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is the natural filtration of this cylindrical Wiener process.*

Before giving the proof let us recall an Itô's formula in Hilbert spaces, cf., e.g. [10, Theorem 4.17, P.105].

Proposition 1.11 *Let $\{x(t), t \in [0, T]\}$ be an K -valued process given by*

$$x(t) = x(0) + \int_0^t b(s) ds + \int_0^t \sigma(s) dW(s),$$

where $b(\cdot) \in L^2_{\mathcal{F}}(0, T; K)$ and $\sigma(\cdot) \in L^2_{\mathcal{F}}(0, T; L_2(H; K))$. Suppose that $\Psi \in C^2(K)$. Then, for each $t \in [0, T]$,

$$\begin{aligned} \Psi(x(t)) &= \Psi(x(0)) + \int_0^t D\Psi(x(s))(b(s)) ds \\ &+ \int_0^t D\Psi(x(s))(\sigma(s) dW(s)) + \frac{1}{2} \int_0^t \text{tr}[D^2\Psi(x(s))\sigma(s)\mathcal{Q}^{\frac{1}{2}}(\sigma(s)\mathcal{Q}^{\frac{1}{2}})^*] ds, \end{aligned}$$

where tr denotes trace. In particular, if $\Psi \equiv |\cdot|_K^2$, then we have a.s.

$$\begin{aligned} |x(t)|_K^2 &= |x(0)|_K^2 + 2 \int_0^t \langle x(s), b(s) \rangle_K ds \\ &+ 2 \int_0^t \langle x(s), \sigma(s) dW(s) \rangle_K + \int_0^t \text{tr}[\sigma(s)\mathcal{Q}^{\frac{1}{2}}(\sigma(s)\mathcal{Q}^{\frac{1}{2}})^*] ds. \end{aligned}$$

Proof of Theorem 1.9. Uniqueness : Suppose that (Y, Z) and (Y', Z') are two solutions of (1.12). From Itô's formula it follows that

$$\begin{aligned}
& |Y(t) - Y'(t)|_K^2 = \\
& 2 \int_t^T \langle Y(s) - Y'(s), f(s, Y(s), Z(s)) - f(s, Y'(s), Z'(s)) \rangle_K ds \\
& - 2 \int_t^T \langle Y(s) - Y'(s), (g(s, Y(s), Z(s)) - g(s, Y'(s), Z'(s))) dW(s) \rangle_K \\
& - \int_t^T \|g(s, Y(s), Z(s)) - g(s, Y'(s), Z'(s))\|_{L_2^Q(H; K)}^2 ds. \tag{1.14}
\end{aligned}$$

We shall suppress writing subscripts under the norms in the rest of this proof.

Note that, from (A2) and (A3), we find that

$$\begin{aligned}
& -|g(s, Y(s), Z(s)) - g(s, Y'(s), Z'(s))|^2 \leq \\
& -|g(s, Y(s), Z(s)) - g(s, Y(s), Z'(s))|^2 \\
& + 2|g(s, Y(s), Z(s)) - g(s, Y(s), Z'(s))| \times \\
& |g(s, Y(s), Z'(s)) - g(s, Y'(s), Z'(s))| \\
& - |g(s, Y(s), Z'(s)) - g(s, Y'(s), Z'(s))|^2 \\
& \leq -\alpha |Z(s) - Z'(s)|^2 + 2k |Z(s) - Z'(s)| |Y(s) - Y'(s)| \\
& - |g(s, Y(s), Z'(s)) - g(s, Y'(s), Z'(s))|^2 \\
& \leq (-\alpha + \epsilon) |Z(s) - Z'(s)|^2 + \frac{k^2}{\epsilon} |Y(s) - Y'(s)|^2,
\end{aligned}$$

for any $\epsilon > 0$. Also, by using (A2), we get

$$\begin{aligned}
& \langle Y(s) - Y'(s), f(s, Y(s), Z(s)) - f(s, Y'(s), Z'(s)) \rangle \\
& \leq \left(\frac{1}{\epsilon} + \epsilon k\right) |Y(s) - Y'(s)|^2 + \epsilon k |Z(s) - Z'(s)|^2, \quad \forall \epsilon > 0.
\end{aligned}$$

Hence, by installing these two latter inequalities in (1.14), we obtain

$$\begin{aligned}
\mathbb{E} |Y(t) - Y'(t)|^2 & \leq \left(\frac{1}{\epsilon} + \epsilon k + \frac{k^2}{\epsilon}\right) \mathbb{E} \int_t^T |Y(s) - Y'(s)|^2 ds \\
& + (-\alpha + \epsilon + \epsilon k) \mathbb{E} \int_t^T |Z(s) - Z'(s)|^2 ds, \tag{1.15}
\end{aligned}$$

$\forall \epsilon > 0$. Therefore, by choosing $\epsilon = \frac{\alpha}{2(1+k)}$, we conclude from Gronwall's inequality that $\mathbb{E} |Y(t) - Y'(t)|^2 = 0$, for all $t \in [0, T]$ and so (1.15) implies that $\mathbb{E} \int_0^T |Z(s) - Z'(s)|^2 ds = 0$.

Existence : We shall divide the proof into two steps.

Step 1 : We study the following simplified version of (1.13),

$$Y(t) = \xi + \int_t^T f(s) ds - \int_t^T g(s, Z(s)) dW(s),$$

where f does not depend on Y and Z , while g depends on Z but not on Y .

In this case define, for each t ,

$$Y(t) := \mathbb{E} \left[\xi + \int_t^T f(s) ds \mid \mathcal{F}_t \right].$$

By making use of Corollary 1.4 there exists a unique $\tilde{Z} \in L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$, such that

$$\mathbb{E} \left[\xi + \int_0^T f(s) ds \mid \mathcal{F}_t \right] = \mathbb{E} \left[\xi + \int_0^T f(s) ds \right] + \int_0^t \tilde{Z}(s) dW(s),$$

which implies that

$$Y(t) = \xi + \int_t^T f(s) ds - \int_t^T \tilde{Z}(s) dW(s).$$

It remains to show that given $\tilde{Z} \in L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$, there exists $Z \in L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$ such that $g(t, Z(t)) = \tilde{Z}(t)$. Since g is bijection in the Z -variable, for any $(t, \omega, z) \in [0, T] \times \Omega \times L^{\mathcal{Q}}_2(H; K)$, according to (A4), there exists a unique $\phi(t, \omega, z) \in L^{\mathcal{Q}}_2(H; K)$, such that $g(t, \phi(t, \omega, z)) = z$. Thus we have to show only that ϕ is $\mathcal{P} \otimes \mathcal{B}(K) \otimes \mathcal{B}(L^{\mathcal{Q}}_2(H; K)) / \mathcal{B}(L^{\mathcal{Q}}_2(H; K))$ measurable. We may assume, without loss of generality, that $\Omega \equiv C_0([0, T]; H)$, $\mathcal{F}_T \equiv \mathcal{B}(\Omega)$, $W(t)(\omega) = \omega(t)$, $\forall t \in [0, T]$.

From the properties of g we see that the mapping $G(t, \omega, z) := (t, \omega, g(t, \omega, z))$, defined from $E \equiv [0, T] \times \Omega \times L^{\mathcal{Q}}_2(H; K)$ into itself is Borel measurable and bijection. Since E is a complete, separable metric space,

then it follows from *Kuratowski's* Theorem, see e.g. [44, Theorem 3.9 and Corollary 3.3], that G^{-1} is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(L_2(H; K))$ measurable. It follows then that the restriction of G to each sub-interval $[0, t]$ for any $t \in [0, T]$, is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t \otimes \mathcal{B}(L_2^{\mathcal{Q}}(H; K))$ measurable. Thereby, ϕ is $\mathcal{P} \otimes \mathcal{B}(K) \otimes \mathcal{B}(L_2^{\mathcal{Q}}(H; K)) / \mathcal{B}(L_2^{\mathcal{Q}}(H; K))$ measurable.

Step 2 : f and g satisfy (A1)–(A4). By making use of Step 1, this case follows directly in the same way the standard result of Pardoux and Peng [41], in the finite dimensional case, was proved. ■

Let us now try to study the following BSDE under some weaker conditions than those mentioned above.

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t)) dt - Z(t) dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases} \quad (1.16)$$

or, in particular, for $0 \leq t \leq T$,

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s) \quad a.s. \quad (1.17)$$

Our new conditions are the following.

- (B1) The mappings $f(\cdot, 0, 0) \in L_{\mathcal{F}}^2(0, T; K)$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$,
- (B2) (Monotonicity condition) $\exists \mu \in \mathbb{R}$ (positive or negative) such that for all $y, y' \in K$

$$\langle f(t, y, z) - f(t, y', z), y - y' \rangle_K \leq \mu |y - y'|_K^2,$$

$\forall z \in L_2^{\mathcal{Q}}(H; K)$, (t, ω) a.e. Note that, for example, if f is Lipschitz in y this condition holds,

- (B3) $\exists k > 0$ such that $\forall z, z' \in L_2^{\mathcal{Q}}(H; K)$

$$|f(t, y, z) - f(t, y, z')|_K^2 \leq k |z - z'|_{L_2^{\mathcal{Q}}(H; K)}^2,$$

$\forall y \in K$, (t, ω) a.e.,

- (B4) $\forall (t, y, z), |f(t, y, z)|_K \leq |\bar{f}(t)| + \kappa (1 + |y|_K + |z|_{L_2^{\mathcal{Q}}(H; K)})$ a.s., for some \mathbb{R}_+ -valued progressively measurable process $\{\bar{f}(t), 0 \leq t \leq T\}$, such that $\mathbb{E} \int_0^T |\bar{f}(t)|^2 < \infty$.
- (B5) The mapping $y \mapsto f(t, y, z)$ is continuous, $\forall t$, a.s.

For a fixed $Z \in L_{\mathcal{F}}^2(0, T; L_2(H; K))$, denote $h(t, y) \triangleq f(t, y, Z(t))$, $(t, y) \in [0, T] \times K$, where f is the mapping mentioned above. Then $h : [0, T] \times \Omega \times K \rightarrow K$ satisfies the following conditions

- (B1)' $\mathbb{E} \int_0^T |h(s, 0)|_K^2 < \infty$,
- (B2)' $\langle y - y', h(t, y) - h(t, y') \rangle_K \leq \mu |y - y'|_K^2$,
- (B4)' $\forall (t, y), |h(t, y)|_K \leq |\bar{f}(t)| + \kappa (1 + |y|_K + |Z(t)|_{L_2^{\mathcal{Q}}(H; K)})$ a.s.,
- (B5)' $y \mapsto h(t, y)$ is continuous, $\forall t$, a.s.

In what follows we demonstrate how we can construct a sequence, $\{h_n\}_{n \geq 1}$, of mappings defined from $[0, T] \times \Omega \times K \rightarrow K$, which satisfy, for each n , (B1)', (B2)', (B4)' (with $\tilde{\kappa} := \kappa + 2|\mu|$ replacing κ) and are Lipschitz in y uniformly in (t, ω) . Furthermore, $h_n(t, \cdot)$ converges pointwisely to $h(t, \cdot)$.

We shall do this by using the *Yosida's approximation* as follows.

Let us first recall the following lemma, which we will need.

Lemma 1.12 *Let $F : K \rightarrow K$ be a continuous function satisfying*

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle_K \leq 0, \quad \forall x_1, x_2 \in K.$$

For $\alpha > 0$, set

$$F_{\alpha}(x) := F(J_{\alpha}(x)) = \frac{1}{\alpha} (J_{\alpha}(x) - x),$$

where $J_{\alpha}(x) := (I - \alpha F)^{-1}(x)$. Then

(i) for any $\alpha > 0$, we have

$$|J_{\alpha}(x_1) - J_{\alpha}(x_2)|_K \leq |x_1 - x_2|_K, \quad \forall x_1, x_2 \in K,$$

(ii) $\lim_{\alpha \rightarrow 0} J_\alpha(x) = x$, $\forall x \in K$,

(iii) for any $\alpha > 0$, we have

$$\langle F_\alpha(x_1) - F_\alpha(x_2) , x_1 - x_2 \rangle_K \leq 0 , \quad \forall x_1, x_2 \in K,$$

(iv) $|F_\alpha(x_1) - F_\alpha(x_2)|_K \leq \frac{2}{\alpha} |x_1 - x_2|_K$, $\forall x_1, x_2 \in K$,

(v) $|F_\alpha(x)|_K \leq |F(x)|_K$, $\forall x \in K$.

The proof of this lemma can be found in [10]. See also [61].

Notice that from (ii) and the continuity of F it follows that

$$\lim_{\alpha \rightarrow 0} F_\alpha(x) = F(x) , \quad \forall x \in K.$$

We now go back to our question of approximating the mapping h above.

Let $F : [0, T] \times \Omega \times K \rightarrow K$ be defined by

$$F(t, y) := h(t, y) - \mu y , \quad (t, y) \in [0, T] \times K.$$

Then F is continuous in y , and satisfies, moreover,

$$\langle F(t, y_1) - F(t, y_2) , y_1 - y_2 \rangle_K \leq 0 , \quad \forall y_1, y_2 \in K,$$

by using (B2)'. Thus by applying Lemma 1.12, we find that the mapping F_α , given by

$$F_\alpha(t, y) := F(J_\alpha(t, y)) , \quad (t, y) \in [0, T] \times K,$$

satisfies the properties (iii)–(v) above.

Now define, for $\alpha > 0$, $h_\alpha : [0, T] \times \Omega \times K \rightarrow K$ by

$$h_\alpha(t, y) := F_\alpha(t, y) + \mu y , \quad (t, y) \in [0, T] \times K.$$

Then it is easily seen that from the properties of F_α that h_α satisfies the following properties:

1. $\langle h_\alpha(t, y_1) - h_\alpha(t, y_2), y_1 - y_2 \rangle_K \leq \mu |y_1 - y_2|_K^2, \quad \forall y_1, y_2 \in K,$
2. $|h_\alpha(t, y_1) - h_\alpha(t, y_2)|_K \leq (\frac{2}{\alpha} + |\mu|) |y_1 - y_2|_K, \quad \forall y_1, y_2 \in K \quad \text{and} \quad \forall (t, \omega),$
3. $|h_\alpha(t, y)|_K \leq |h(t, y)|_K + 2|\mu| |y|_K, \quad \forall y \in K, \quad \text{and}$
4. $\lim_{\alpha \rightarrow 0} h_\alpha(t, y) = h(t, y), \quad \forall y \in K.$

Therefore h_α satisfies the required conditions of approximation; if necessary let $\alpha := \frac{1}{n}$. In particular, we have proved the following proposition.

Proposition 1.13 *There exists a sequence sequence, $\{h_n\}_{n \geq 1}$, of mappings, defined from $[0, T] \times \Omega \times K$ to K , which satisfy, for each n , $(B1)'$, $(B2)'$, $(B4)'$ (with $\tilde{\kappa} := \kappa + 2|\mu|$ replacing κ) and are Lipschitz in y uniformly in (t, ω) .*

Moreover, $h_n(t, \cdot)$ converges pointwisely to $h(t, \cdot)$.

The rest of this section will be devoted to proving the following theorem.

Theorem 1.14 *Suppose that $(B1)$ – $(B5)$ hold. There exists a unique strong solution (Y, Z) to BSDE (1.16) in $L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$. Moreover, the process Y has a continuous modification and satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} |Y(t)|_K^2 < \infty. \quad (1.18)$$

This theorem generalises a similar result in finite dimensions of Darling and Pardoux in [12, Theorem 1.2]; see also [43, Theorem 1.3]. We will give a detailed proof here, following the proof scheme of [12], for the sake of completeness.

Proof of Theorem 1.14. Uniqueness : Suppose that (Y, Z) and (Y', Z') are two solutions of (1.17). From Itô's formula it follows that

$$\begin{aligned}
|Y(t) - Y'(t)|_K^2 = & \\
& 2 \int_t^T \langle Y(s) - Y'(s), f(s, Y(s), Z(s)) - f(s, Y'(s), Z(s)) \rangle_K ds \\
& + 2 \int_t^T \langle Y(s) - Y'(s), f(s, Y'(s), Z(s)) - f(s, Y'(s), Z'(s)) \rangle_K ds \\
& - 2 \int_t^T \langle Y(s) - Y'(s), (Z(s) - Z'(s)) dW(s) \rangle_K \\
& - \int_t^T |Z(s) - Z'(s)|_{L_2^{\mathcal{Q}}(H;K)}^2 ds.
\end{aligned} \tag{1.19}$$

By using (B3), we get

$$\begin{aligned}
2 \langle Y(s) - Y'(s), f(s, Y'(s), Z(s)) - f(s, Y'(s), Z'(s)) \rangle_K \\
\leq 2\sqrt{k} |Y(s) - Y'(s)|_K |Z(s) - Z'(s)|_{L_2^{\mathcal{Q}}(H;K)} \\
\leq 2k |Y(s) - Y'(s)|^2 + \frac{1}{2} |Z(s) - Z'(s)|_{L_2^{\mathcal{Q}}(H;K)}^2.
\end{aligned}$$

By substituting this inequality in (1.19), taking expectation and using (B2), we obtain

$$\begin{aligned}
\mathbb{E} |Y(t) - Y'(t)|_K^2 \leq 2 (\mu + k) \mathbb{E} \int_t^T |Y(s) - Y'(s)|_K^2 ds \\
- \frac{1}{2} \mathbb{E} \int_t^T |Z(s) - Z'(s)|_{L_2^{\mathcal{Q}}(H;K)}^2 ds.
\end{aligned} \tag{1.20}$$

The result then follows by using Gronwall's inequality as done in the proof of the preceding theorem.

Existence : Note that (Y, Z) solves (1.17) if and only if $\{(\hat{Y}(t), \hat{Z}(t)) := (e^{\lambda t} Y(t), e^{\lambda t} Z(t)), t \in [0, T]\}$ solves the following BSDE:

$$\begin{aligned}
\hat{Y}(t) &= e^{\lambda T} \xi + \int_t^T [e^{\lambda s} f(s, e^{-\lambda s} \hat{Y}(s), e^{-\lambda s} \hat{Z}(s)) - \lambda \hat{Y}(s)] ds \\
&- \int_t^T \hat{Z}(s) dW(s), \quad 0 \leq t \leq T,
\end{aligned}$$

By letting $\lambda = -\mu$, we find that

$$\hat{f}(s, y, z) := e^{-\mu s} f(s, e^{\mu s} y, e^{\mu s} z) + \mu y,$$

satisfies assumption (B2) with $\mu = 0$, in addition to the rest of conditions (B1) and (B3)–(B5). Hence we may and we will assume until the end of this proof that f satisfies condition (B2) with $\mu = 0$.

Proposition 1.15 *Suppose that assumptions (B1)–(B5) are imposed with $\mu = 0$. Given an $L_2^Q(H; K)$ -valued progressively measurable process $\{V(t), 0 \leq t \leq T\}$ which satisfies $\mathbb{E} \int_0^T |V(s)|_{L_2^Q(H; K)}^2 dt < \infty$, there exists a unique pair $\{(Y(t), Z(t)), 0 \leq t \leq T\}$ of measurable processes with values in $K \times L_2^Q(H; K)$ satisfying*

$$\mathbb{E} \sup_{t \in [0, T]} |Y(t)|_K^2 + \mathbb{E} \int_0^T |Z(t)|_{L_2^Q(H; K)}^2 dt < \infty, \quad (1.21)$$

and

$$Y(t) = \xi + \int_t^T f(s, Y(s), V(s)) ds - \int_t^T Z(s) dW(s), \quad 0 \leq t \leq T. \quad (1.22)$$

The proof of this proposition will be given below.

With the help of Proposition 1.15 we can construct a mapping Φ from $\mathcal{B}^2 \equiv L_{\mathcal{F}}^2(0, T; K) \times L_{\mathcal{F}}^2(0, T; L_2^Q(H; K))$ into itself as follows. For any $(U, V) \in \mathcal{B}^2$, define $\Phi(U, V) \triangleq (Y, Z)$, where (Y, Z) is the unique solution of

$$Y(t) = \xi + \int_t^T f(s, Y(s), V(s)) ds - \int_t^T Z(s) dW(s), \quad 0 \leq t \leq T.$$

Let $(U, V), (U', V') \in \mathcal{B}^2$, $(Y, Z) = \Phi(U, V)$ and $(Y', Z') = \Phi(U', V')$. Denote by $(\bar{U}, \bar{V}) = (U - U', V - V')$, $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. By using Itô's

formula together with (B2) (with $\mu = 0$) , (B3) and (B5) we derive

$$\begin{aligned}
& e^{\gamma t} \mathbb{E} |\bar{Y}(t)|_K^2 + \gamma \mathbb{E} \int_t^T e^{\gamma s} |\bar{Y}(s)|_K^2 ds = \\
& 2 \mathbb{E} \int_t^T e^{\gamma s} \langle \bar{Y}(s), f(s, Y(s), V(s)) - f(s, Y'(s), V(s)) \rangle_K ds \\
& + 2 \mathbb{E} \int_t^T e^{\gamma s} \langle \bar{Y}(s), f(s, Y'(s), V(s)) - f(s, Y'(s), V'(s)) \rangle_K ds \\
& - \mathbb{E} \int_t^T e^{\gamma s} |\bar{Z}(s)|_{L_2^{\mathcal{Q}}(H;K)}^2 ds \\
& \leq 2k \mathbb{E} \int_t^T e^{\gamma s} |\bar{Y}(s)|_K^2 ds + \frac{1}{2} \mathbb{E} \int_t^T e^{\gamma s} |\bar{V}(s)|_{L_2^{\mathcal{Q}}(H;K)}^2 ds \\
& - \int_t^T e^{\gamma s} |\bar{Z}(s)|_{L_2^{\mathcal{Q}}(H;K)}^2 ds
\end{aligned}$$

Therefore, by choosing $\gamma \leq 2k + 1$, we obtain

$$\begin{aligned}
& e^{\gamma t} \mathbb{E} |\bar{Y}(t)|_K^2 + (\gamma - 2k) \mathbb{E} \int_t^T e^{\gamma s} (|\bar{Y}(s)|_K^2 + |\bar{Z}(s)|_{L_2^{\mathcal{Q}}(H;K)}^2) ds \\
& \leq \frac{1}{2} \mathbb{E} \int_t^T e^{\gamma s} (|\bar{U}(s)|_{L_2^{\mathcal{Q}}(H;K)}^2 + |\bar{V}(s)|_{L_2^{\mathcal{Q}}(H;K)}^2) ds.
\end{aligned}$$

This implies that Φ is a strict contraction mapping on the space \mathcal{B}^2 , equipped with the norm

$$\|(Y, Z)\|_{\mathcal{B}^2} := (\mathbb{E} \int_0^T e^{\gamma_0 s} (|Y(s)|_K^2 + |Z(s)|_{L_2^{\mathcal{Q}}(H;K)}^2) ds)^{1/2},$$

where $\gamma_0 := 2k + 1$. Consequently, it has a unique fixed point, (Y, Z) say. It then easy to check that this fixed point (Y, Z) satisfies (1.17).

Finally, we show (1.18). Note that, for each $t \in [0, T]$, $Y(t) = \mathbb{E} [\xi + \int_t^T f(s, Y(s), Z(s)) ds \mid \mathcal{F}_t]$. Thus we have $|Y(t)|_K \leq M(t)$, a.s., for all $t \in [0, T]$, where $M(t) := \mathbb{E} [|\xi|_K + \int_0^T |f(s, Y(s), Z(s))|_K ds \mid \mathcal{F}_t]$. Then, by using Burkholder-Davis-Gundy inequality (or Doob's inequality), (B4) and

(B1) we obtain

$$\begin{aligned}
\mathbb{E} \sup_{t \in [0, T]} |Y(t)|_K^2 &\leq \mathbb{E} \sup_{t \in [0, T]} (M(t))^2 \\
&\leq 4 \mathbb{E} (|\xi| + \int_0^T |f(s, Y(s), Z(s))|_K ds)^2 \\
&\leq 8 \mathbb{E} (|\xi|^2 + T \int_0^T |f(s, Y(s), Z(s))|_K^2 ds) \\
&\leq 8 \mathbb{E} |\xi|^2 + 16 T \mathbb{E} \int_0^T |f(s, 0, 0)|_K^2 ds \\
&\quad + 16 \kappa T \mathbb{E} \int_0^T |Y(s)|_K^2 ds + 16 \kappa^2 T^2 \\
&< +\infty.
\end{aligned}$$

The proof of Theorem 1.14 finishes here. ■

We now prove Proposition 1.15.

Proof of Proposition 1.15. The proof of uniqueness is done as in Theorem 1.14 above.

We now show the existence of a solution to (1.22).

Let us define $h(s, y) \triangleq f(s, y, V(s))$. Then h satisfies (B1)', (B2)', (B4)' and (B5)', mentioned in Proposition 1.13, with V replacing Z . Consequently, by using Proposition 1.13, we observe that, for each n , there exists $h_n : [0, T] \times \Omega \times K \rightarrow K$, which satisfies (B1)', (B2)', (B4)' (with $\tilde{\kappa}$ replacing κ), is Lipschitz in y , uniformly in (t, ω) , and moreover, the following holds

$$\lim_{n \rightarrow \infty} \mathbb{E} \int_0^T |h_n(s, y(s)) - h(s, y(s))|_K^2 ds \rightarrow 0, \quad (1.23)$$

for each $y \in L_{\mathcal{F}}^2(0, T; K)$.

For $n \geq 1$. Consider the following BSDE

$$Y_n(t) = \xi + \int_t^T h_n(s, Y_n(s)) ds - \int_t^T Z_n(s) dW(s), \quad 0 \leq t \leq T. \quad (1.24)$$

By Theorem 1.9 there exists a unique solution $(Y_n, Z_n) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L^{\mathcal{Q}}_2(H; K))$, to (1.24). Furthermore, the following holds by Itô's formula

$$\begin{aligned} |Y_n(t)|_K^2 &= |\xi|_K^2 + 2 \int_t^T \langle Y_n(s), h_n(s, Y_n(s)) \rangle_K ds \\ &\quad - 2 \int_t^T \langle Y_n, Z_n(s) dW(s) \rangle_K - \int_t^T |Z_n(s)|_{L^{\mathcal{Q}}_2(H; K)}^2 ds. \end{aligned} \quad (1.25)$$

Let us now find an estimate for $\mathbb{E} \left(\sup_{t \in [0, T]} |Y_n(t)|_K^2 + \int_0^T |Z_n(s)|_{L^{\mathcal{Q}}_2(H; K)}^2 ds \right)$.

By using (B2)' and (B4)' (for h_n), we conclude that

$$\begin{aligned} 2 \langle Y_n(s), h_n(s, Y_n(s)) \rangle_K &\leq 2 \langle Y_n(s), h_n(s, 0) \rangle_K \\ &\leq \epsilon |Y_n(s)|_K^2 + \frac{2}{\epsilon} |\bar{f}(s)|^2 + \frac{4\kappa^2}{\epsilon} (1 + |V(s)|_{L^{\mathcal{Q}}_2(H; K)}^2), \end{aligned} \quad (1.26)$$

for any $\epsilon > 0$. Therefore, by choosing $\epsilon := \frac{1}{2}$ and taking expectation in (1.25), we obtain

$$\begin{aligned} &\mathbb{E} |Y_n(t)|_K^2 + \mathbb{E} \int_t^T |Z_n(s)|_{L^{\mathcal{Q}}_2(H; K)}^2 ds \\ &\leq \mathbb{E} |\xi|_K^2 + 4 \mathbb{E} \int_t^T |\bar{f}(s)|^2 ds + \frac{1}{2} \mathbb{E} \int_t^T |Y_n(s)|_K^2 ds \\ &\quad + 8\kappa^2 (T - t) + 8\kappa^2 \mathbb{E} \int_t^T |V(s)|_{L^{\mathcal{Q}}_2(H; K)}^2 ds. \end{aligned} \quad (1.27)$$

Hence, by Gronwall's inequality, for each $t \in [0, T]$,

$$\mathbb{E} |Y_n(t)|_K^2 \leq C' e^{\frac{1}{2}(T-t)}, \quad (1.28)$$

where

$$C' := \mathbb{E} |\xi|_K^2 + 4 \mathbb{E} \int_0^T |\bar{f}(s)|^2 ds + 8\kappa^2 T + 8\kappa^2 \mathbb{E} \int_0^T |V(s)|_{L^{\mathcal{Q}}_2(H; K)}^2 ds,$$

which together with (1.27) implies the following

$$\mathbb{E} \int_0^T |Z_n(s)|_{L^{\mathcal{Q}}_2(H; K)}^2 ds \leq C' e^{\frac{T}{2}}.$$

Thus, in particular,

$$\sup_{n \geq 1} \mathbb{E} |Y_n(t)|_K^2 < \infty \quad (1.29)$$

and

$$\sup_{n \geq 1} \mathbb{E} \int_0^T |Z_n(s)|_{L_2^{\mathcal{Q}}(H;K)}^2 ds < \infty. \quad (1.30)$$

Moreover by using the same method used for proving (1.18), as (B4)' and (B1)' hold here, together with (1.29) and (1.30) we deduce, in particular, that

$$\sup_{n \geq 1} \mathbb{E} \left(\sup_{t \in [0, T]} |Y_n(t)|_K^2 \right) < \infty. \quad (1.31)$$

For $s \in [0, T]$, define $U_n(s) := h_n(s, Y_n(s))$. Then from (B4)', (1.31) and (A1) we find that

$$\sup_{n \geq 1} \mathbb{E} \int_0^T |U_n(s)|_K^2 ds < \infty. \quad (1.32)$$

Hence there exist subsequences $\{(Y_{n_k}, Z_{n_k}, U_{n_k}), k \geq 1\}$, which converges weakly in $L_{\mathcal{F}}^2(0, T; K) \times L_{\mathcal{F}}^2(0, T; L_2^{\mathcal{Q}}(H; K)) \times L_{\mathcal{F}}^2(0, T; K)$. Call their weak limits (Y, Z, U) .

Let η be an arbitrary element of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$. As a consequence of Corollary 1.4 we find that any such η is written in the following way

$$\eta = \mathbb{E} \eta + \int_0^T Z^\eta(s) dW(s),$$

for some $Z^\eta \in L_{\mathcal{F}}^2(0, T; L_2^{\mathcal{Q}}(H; K))$. Hence

$$\begin{aligned} \mathbb{E} \left\langle \eta, \int_0^T Z_{n_k}(s) dW(s) \right\rangle_K &= \mathbb{E} \int_0^T \langle Z^\eta(s), Z_{n_k}(s) \rangle_{L_2^{\mathcal{Q}}(H;K)} ds \\ &\rightarrow \mathbb{E} \int_0^T \langle Z^\eta(s), Z(s) \rangle_{L_2^{\mathcal{Q}}(H;K)} ds \\ &= \mathbb{E} \left\langle \eta, \int_0^T Z(s) dW(s) \right\rangle_K. \end{aligned} \quad (1.33)$$

I.e. $\int_0^T Z_{n_k}(s) dW(s) \rightarrow \int_0^T Z(s) dW(s)$ weakly in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$, as $k \rightarrow \infty$.

Denote, with abuse of notation, $\eta(t) := \mathbb{E} [\eta | \mathcal{F}_t]$, $0 \leq t \leq T$. We then observe, similarly, that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \langle \eta(\cdot), \int_0^\cdot Z_{n_k}(s) dW(s) \rangle_{L^2_{\mathcal{F}}(0,T;K)} \\
&= \lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle \eta(s), \int_0^s Z_{n_k}(r) dW(r) \rangle_K ds \\
&= \lim_{k \rightarrow \infty} \int_0^T (\mathbb{E} \int_0^T \langle \chi_{[0,s]}(r) \cdot Z^\eta(r), Z_{n_k}(r) \rangle_{L^2_2(H;K)} dr) ds \\
&= \lim_{k \rightarrow \infty} \int_0^T \mathbb{E} \langle \eta(s), \int_0^s Z(r) dW(r) \rangle_K ds \\
&= \langle \eta(\cdot), \int_0^\cdot Z(s) dW(s) \rangle_{L^2_{\mathcal{F}}(0,T;K)}. \tag{1.34}
\end{aligned}$$

We have used here the dominated convergence theorem. By using a similar argument one can easily see that (1.34) also holds for arbitrary element η of $L^2_{\mathcal{F}}(0,T;K)$. Thus, in particular, $\int_0^\cdot Z_{n_k} dW(s) \rightarrow \int_0^\cdot Z(s) dW(s)$ weakly in $L^2_{\mathcal{F}}(0,T;K)$, as $k \rightarrow \infty$.

Similarly for $\int_t^T Z(s) dW(s) = \int_0^T Z(s) dW(s) - \int_0^t Z(s) dW(s)$ and so we conclude that $\int_t^T Z_{n_k} dW(s) \rightarrow \int_t^T Z(s) dW(s)$ weakly in $L^2_{\mathcal{F}}(0,T;K)$, as $k \rightarrow \infty$. This can be seen easily from (1.34) and the following result.

$$\begin{aligned}
& \mathbb{E} \int_0^T \langle \eta(s), \int_0^T Z_{n_k}(r) dW(r) \rangle_K ds \rightarrow \\
& \mathbb{E} \int_0^T \langle \eta(s), \int_0^T Z(r) dW(r) \rangle_K ds,
\end{aligned}$$

as $k \rightarrow \infty$, by the dominated convergence theorem and the first case (1.33).

Eventually, note that if we pass the weak limit as $n \rightarrow \infty$ into (1.24), we derive the following BSDE

$$Y(t) = \xi + \int_t^T U(s) ds - \int_t^T Z(s) dW(s), \quad 0 \leq t \leq T.$$

The proof then finishes once we prove that $U(s) = h(s, Y(s))$.

Although our setting here is infinite dimensional, we can still follow the same idea as in [12]. Let $\{X(t), 0 \leq t \leq T\}$ be any element of $L^2_{\mathcal{F}}(0, T; K)$. Then by (B2)' (for h_{n_k}), we obtain

$$\mathbb{E} \int_0^T \langle Y_{n_k}(s) - X(s), h_{n_k}(s, Y_{n_k}(s)) - h_{n_k}(s, X(s)) \rangle_K ds \leq 0 .$$

This together with (1.23) and (1.28) implies that

$$\limsup_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle Y_{n_k}(s) - X(s), h_{n_k}(s, Y_{n_k}(s)) - h(s, X(s)) \rangle_K ds \leq 0. \quad (1.35)$$

Moreover,

$$\begin{aligned} 2 \mathbb{E} \int_0^T \langle Y_{n_k}(s), h_{n_k}(s, Y_{n_k}(s)) \rangle_K ds = \\ |Y_{n_k}(0)|_K^2 - \mathbb{E} |\xi|_K^2 + \mathbb{E} \int_0^T |Z_{n_k}(s)|_{L^2_{\mathcal{Q}}(H; K)}^2 ds . \end{aligned}$$

Since $Y_{n_k}(0)$ is deterministic, as it is \mathcal{F}_0 -measurable, $Y_{n_k}(0) \rightarrow Y(0)$ in K , as $k \rightarrow \infty$.

On the other hand, since the real-valued mapping

$$Z \mapsto \mathbb{E} \int_0^T |Z(s)|_{L^2_{\mathcal{Q}}(H; K)}^2 ds$$

is convex and continuous on $L^2_{\mathcal{F}}(0, T; L^2_{\mathcal{Q}}(H; K))$ with respect to the strong topologies, it is lower semi-continuous with respect to the weak topologies, see [51, Theorem S.7, P. 356] . This implies that

$$\begin{aligned} \liminf_{k \rightarrow \infty} 2 \mathbb{E} \int_0^T \langle Y_{n_k}(s), h_{n_k}(s, Y_{n_k}(s)) \rangle_K ds \geq \\ |Y(0)|_K^2 - \mathbb{E} |\xi|_K^2 + \mathbb{E} \int_0^T |Z(s)|_{L^2_{\mathcal{Q}}(H; K)}^2 ds \\ = 2 \mathbb{E} \int_0^T \langle Y(s), U(s) \rangle_K ds . \end{aligned} \quad (1.36)$$

Hence, from (1.35), (1.36) and from the weak convergence of Y_{n_k} and U_{n_k} , we get that

$$\begin{aligned}
0 &\geq \liminf_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle Y_{n_k}(s) - X(s), h_{n_k}(s, Y_{n_k}(s)) - h(s, X(s)) \rangle_K ds \\
&= \liminf_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle Y_{n_k}(s), h_{n_k}(s, Y_{n_k}(s)) \rangle_K ds \\
&\quad - \lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle Y_{n_k}(s), h(s, X(s)) \rangle_K ds \\
&\quad - \lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle X(s), h_{n_k}(s, Y_{n_k}(s)) \rangle_K ds \\
&\quad + \mathbb{E} \int_0^T \langle X(s), h(s, X(s)) \rangle_K ds \\
&\geq \mathbb{E} \int_0^T \langle Y(s), U(s) \rangle_K ds - \mathbb{E} \int_0^T \langle Y(s), h(s, X(s)) \rangle_K ds \\
&\quad - \mathbb{E} \int_0^T \langle X(s), U(s) \rangle_K ds + \mathbb{E} \int_0^T \langle X(s), h(s, X(s)) \rangle_K ds \\
&= \mathbb{E} \int_0^T \langle Y(s) - X(s), U(s) - h(s, X(s)) \rangle_K ds. \tag{1.37}
\end{aligned}$$

Choose, for $0 \leq s \leq T$, $X(s) \triangleq Y(s) - \epsilon [U(s) - h(s, Y(s))]$, for some $\epsilon > 0$, in (1.37). Then divide by ϵ and let $\epsilon \rightarrow 0$ to obtain eventually the following result

$$\mathbb{E} \int_0^T |U(s) - h(s, Y(s))|_K^2 ds = 0,$$

as required. ■

Remark 1.16 (i) Unfortunately, we do not know if one could weaken assumptions (A1)–(A4) for the BSDE (1.12) as we have done for the BSDE (1.16) and get the same result as in Theorem 1.14. The uniqueness proof goes well without problems, but the problem we face is with the existence proof. This is due to the weak convergence used in the preceding proof which fails to work in general when having a general g even when g depends linearly on z , i.e. of the type $g(s, y) + z$.

In Chapter 2 we give another type of non-global Lipschitz condition, see remark (ii) below, under which it is possible to achieve results similar to those obtained earlier for such a general equation, though we will not discuss that explicitly as we will concentrate on more general equations, the so-called, backward stochastic evolution equations, which makes the proof totally different from the familiar one.

(ii) Note that one can replace assumption (A2) by the following one

$$|f(t, y, z) - f(t, y', z')|_K^2 \leq c(|y - y'|_K^2) + k |z - z'|_{L_2^{\mathcal{Q}}(H; K)}^2$$

and

$$|g(t, y, z) - g(t, y', z')|_{L_2^{\mathcal{Q}}(H; K)}^2 \leq c(|y - y'|_K^2) + k |z - z'|_{L_2^{\mathcal{Q}}(H; K)}^2 ,$$

where c is as in assumption (C2) in Chapter 2 below, to get exactly the same result as in Theorem 1.14 above. The proof works well, essentially, as in the finite dimensional case which was done by Mao in [33]. For this case, we will need to use Theorem 1.9 above.

(iii) Theorem 1.14 still holds when W is replaced by cylindrical Wiener process on H . In this case $Z \in L_{\mathcal{F}}^2(0, T; L_2(H; K))$; cf. Remark 1.10

1.4 Applications of Backward Stochastic Differential Equations

In the previous section we established the existence and uniqueness of the solution for BSDEs under non-global Lipschitz conditions. In this section let the terminal value ξ , appearing in the BSDE (1.16), be of the form $g(X^{t,x}(T))$, where $\{X^{t,x}(s), t \leq s \leq T\}$ is a diffusion in H , starting from x at time $s = t$ and $g : H \rightarrow K$ is a nice continuous map. We shall show

that the corresponding solution $Y^{t,x}(s)$, $t \leq s \leq T$, to the following BSDE

$$\begin{aligned} Y^{t,x}(s) = & g(X^{t,x}(T)) + \int_s^T f(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r)) dr \\ & - \int_s^T Z^{t,x}(r) dW(r), \end{aligned} \quad (1.38)$$

depends continuously in L^2 on x .

For this sort of BSDEs we shall provide a comparison theorem between the solutions.

On the other hand, we shall give a representation to the solutions of a system of semi-linear parabolic PDEs of type

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), Du(t, x) \sigma(t, x)) = 0, \\ u(T, x) = g(x), \end{cases} \quad (1.39)$$

by using the solution of the corresponding BSDE of the type (1.38). The operator \mathcal{L} is defined in (1.51) below. This establishes a uniqueness property of the solution of (1.39). Under weaker conditions than smoothness of the coefficients of both the BSDE (1.38) and (1.39), and in particular under conditions which guarantee that (1.38) has a unique solution in the usual sense, we prove that the mapping $u(t, x) := Y^{t,x}(t)$ is a viscosity solution to the system of PDEs (1.39). Precisely, we deal first with case having a finite dimensional noise, e.g. when $\dim H < \infty$. In another case, without this restriction, we require that f does not depend on the derivative of u , hence not on the variable Z . In which case, we prove that $u(t, x) := Y^{t,x}(t)$ is a viscosity solution to the system of PDEs (1.39) (with f does depend on $Du \sigma$). The comparison theorem and the strict comparison theorem play a major role here.

Let W be a \mathcal{Q} -Wiener process in H . Let $b : [0, T] \times H \rightarrow H$, $\sigma : [0, T] \times H \rightarrow L_2^{\mathcal{Q}}(H; H)$ be two measurable mappings, which are globally Lipschitz with respect to the space variable, x , uniformly in t , and satisfy a linear growth condition in x , uniformly in t .

Consider the following SDE

$$\begin{cases} dX^{t,x}(s) = b(s, X^{t,x}(s))ds + \sigma(s, X^{t,x}(s)) dW(s), & t \leq s \leq T, \\ X^{t,x}(t) = x. \end{cases} \quad (1.40)$$

It is known that (1.40) has a unique solution $\{X^{t,x}(s) \mid t \leq s \leq T\}$, adapted to $\mathcal{F}_s^t = \sigma\{W(r) - W(t), \mid t \leq r \leq s\} \vee \mathcal{N}$, each s , where \mathcal{N} is the collection of \mathbb{P} -null sets, and in L^p for all $1 \leq p < \infty$; see e.g [19].

Consider the following BSDE

$$\begin{aligned} Y^{t,x}(s) &= g(X^{t,x}(T)) + \int_s^T f(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r)) dr \\ &\quad - \int_s^T Z^{t,x}(r) dW(r), \end{aligned} \quad (1.41)$$

where $f : [0, T] \times H \times K \times L_2^{\mathcal{Q}}(H; K) \rightarrow L_2^{\mathcal{Q}}(H; K)$ and $g : H \rightarrow K$, are continuous and satisfy

- (B1)" $|g(x)|_K^2 \leq k(1 + |x|^p)^2$, all $x \in K$,
- (B2)" $\forall y, y' \in K$

$$\langle f(t, x, y, z) - f(t, x, y', z), y - y' \rangle_K \leq \mu |y - y'|_K^2,$$

$$\forall x \in H \text{ and } \forall z \in L_2^{\mathcal{Q}}(H; K), (t, \omega) \text{ a.e.}$$

- (B3)" $\forall z, z' \in L_2^{\mathcal{Q}}(H; K)$

$$|f(t, x, y, z) - f(t, x, y, z')|_K^2 \leq k |z - z'|_{L_2^{\mathcal{Q}}(H; K)}^2,$$

$$\forall y \in K, \forall x \in H, (t, \omega) \text{ a.e.}$$

- (B4)" $|f(t, x, y, z)|_K^2 \leq k(1 + |x|_H^p + |y|_K + |z|_{L_2^{\mathcal{Q}}(H; K)})^2$,

for some positive constants k , p and μ . It follows from Theorem 1.14 that (1.41) has a unique solution $(Y, Z) \in L_{\mathcal{F}^t}^2(t, T; K) \times L_{\mathcal{F}^t}^2(t, T; L_2^{\mathcal{Q}}(H; K))$.

Define, respectively, $X^{t,x}(s), Y^{t,x}(s)$ for any $(t, s, x) \in [0, T]^2 \times H$ by $X^{t,x}(s \vee t), Y^{t,x}(s \vee t)$ and let $Z^{t,x}(s) = 0$ if $s < t$.

It is not hard to prove the following result: $[0, T]^2 \times H \ni (t, s, x) \mapsto X^{t,x}(s)$ is continuous into L^2 with respect to the three variables and moreover

$$\mathbb{E} \sup_{s \in [0, T]} |X^{t',x'}(s) - X^{t,x}(s)|_H^2 \leq C (|x' - x|_H^2 + (1 + |x|_H^2)|t' - t|), \quad (1.42)$$

for some constant $C > 0$, that depends only on b, σ and T . See e.g. [10].

Let us now estimate the difference between the following two solution of the following two BSDEs

$$\begin{aligned} Y_i^{t_i, x_i}(s) &= g_i(X^{t_i, x_i}(T)) + \int_s^T f_i(r, X^{t_i, x_i}(r), Y_i^{t_i, x_i}(r), Z_i^{t_i, x_i}(r)) dr \\ &\quad - \int_s^T Z_i^{t_i, x_i}(r) dW(r), \quad 0 \leq s \leq T, \quad i = 1, 2, \end{aligned} \quad (1.43)$$

where f_i and g_i are continuous and satisfy (B1)"– (B4)", for all $i = 1, 2$. Under these conditions we claim the following proposition.

Proposition 1.17

$$\begin{aligned} &\mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_1^{t_1, x_1}(s) - Y_2^{t_2, x_2}(s)|_K^2 + \int_0^T |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|_{L_2^2(H; K)}^2 dr \right) \\ &\leq C \left(\mathbb{E} |g_1(X^{t_1, x_1}(T)) - g_2(X^{t_2, x_2}(T))|_K^2 + \mathbb{E} \int_0^T |\delta_2 f(r)|_K^2 dr \right), \end{aligned} \quad (1.44)$$

where, for $r \in [0, T]$, $\delta_2 f(r) \triangleq f_1(r, X^{t_1, x_1}(r), Y_2^{t_2, x_2}(r), Z_2^{t_2, x_2}(r)) - f_2(r, X^{t_2, x_2}(r), Y_2^{t_2, x_2}(r), Z_2^{t_2, x_2}(r))$, and $C > 0$, some constant depending only on μ, k and T .

Proof. Apply Itô's formula to the difference $Y_1^{t_1, x_1}(s) - Y_2^{t_2, x_2}(s), t \leq s \leq T$,

as we did for (1.19), to get

$$\begin{aligned}
& |Y_1^{t_1, x_1}(s) - Y_2^{t_2, x_2}(s)|_K^2 = |g_1(X^{t_1, x_1}(T)) - g_2(X^{t_2, x_2}(T))|_K^2 \\
& + 2 \int_s^T \langle Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r), \delta f(r) \rangle_K dr \\
& - 2 \int_s^T \langle Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r), (Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)) dW(r) \rangle_K \\
& - \int_s^T |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|_{L_2^{\mathcal{Q}}(H; K)}^2 dr \\
& \leq |g_1(X^{t_1, x_1}(T)) - g_2(X^{t_2, x_2}(T))|_K^2 \\
& + 2(\mu + 1 + k) \int_s^T |Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r)|_K^2 dr \\
& + \frac{1}{2} \int_s^T |\delta_2 f(r)|_K^2 dr - \frac{1}{2} \int_s^T |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|_{L_2^{\mathcal{Q}}(H; K)}^2 dr \\
& - 2 \int_s^T \langle Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r), (Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)) dW(r) \rangle_K, \quad (1.45)
\end{aligned}$$

where

$$\begin{aligned}
\delta f(r) & \triangleq f_1(r, X^{t_1, x_1}(r), Y_1^{t_1, x_1}(r), Z_1^{t_1, x_1}(r)) \\
& - f_2(r, X^{t_2, x_2}(r), Y_2^{t_2, x_2}(r), Z_2^{t_2, x_2}(r)).
\end{aligned}$$

Moreover, by taking expectation in (1.45) and using Gronwall's inequality, we obtain

$$\begin{aligned}
& \mathbb{E} |Y_1^{t_1, x_1}(s) - Y_2^{t_2, x_2}(s)|_K^2 \leq e^{2(\mu+1+k)(T-s)} \times \\
& (\mathbb{E} |g_1(X^{t_1, x_1}(T)) - g_2(X^{t_2, x_2}(T))|_K^2 + \frac{1}{2} \mathbb{E} \int_s^T |\delta_2 f(r)|_K^2 dr) \quad (1.46)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \int_s^T |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|_{L_2^{\mathcal{Q}}(H; K)}^2 dr \leq 2 e^{2(\mu+1+k)(T-s)} \times \\
& (\mathbb{E} |g_1(X^{t_1, x_1}(T)) - g_2(X^{t_2, x_2}(T))|_K^2 + \frac{1}{2} \mathbb{E} \int_s^T |\delta_2 f(r)|_K^2 dr), \quad (1.47)
\end{aligned}$$

$\forall s \in [t, T]$. Going back to (1.45), suppressing the norms subscript, we conclude that

$$\begin{aligned}
& \mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_1^{t_1, x_1}(s) - Y_2^{t_2, x_2}(s)|^2 + \frac{1}{2} \int_0^T |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|^2 dr \right) \\
& \leq \mathbb{E} |g_1(X^{t_1, x_1}(T)) - g_2(X^{t_2, x_2}(T))|^2 \\
& + 2(\mu + 1 + k) \mathbb{E} \int_0^T |Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r)|^2 dr + \frac{1}{2} \mathbb{E} \int_0^T |\delta_2 f(r)|^2 dr \\
& + 2 \mathbb{E} \sup_s \left[- \int_s^T \langle Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r), (Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)) dW(r) \rangle \right]. \quad (1.48)
\end{aligned}$$

Moreover, by using Burkholder-Davis-Gundy inequality, it follows that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq T} \left[- \int_s^T \langle Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r), (Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)) dW(r) \rangle \right] \\
& = \mathbb{E} \sup_{0 \leq s \leq T} \int_0^s \langle Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r), (Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)) dW(r) \rangle \\
& \leq 4\sqrt{2} \mathbb{E} \left(\int_0^T |Y_1^{t_1, x_1}(r) - Y_2^{t_2, x_2}(r)|^2 |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|^2 dr \right)^{1/2} \\
& \leq 4\sqrt{2} \mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_1^{t_1, x_1}(s) - Y_2^{t_2, x_2}(s)|^2 \int_0^T |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|^2 dr \right)^{1/2} \\
& \leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq T} |Y_1^{t_1, x_1}(s) - Y_2^{t_2, x_2}(s)|^2 \\
& + 32 \mathbb{E} \int_0^T |Z_1^{t_1, x_1}(r) - Z_2^{t_2, x_2}(r)|^2 dr. \quad (1.49)
\end{aligned}$$

Finally, by substituting (1.49), (1.46) and (1.47) into (1.48), we derive our claim (1.44). ■

We now proceed to prove one of the main important tools of the BSDEs, the so-called, comparison theorem, in the case $K \equiv \mathbb{R}$.

Theorem 1.18 *Let f_i and g_i be as in Proposition 1.17 with the corresponding solutions $(Y_1^{t, x}(s), Z_1^{t, x}(s))$, $(Y_2^{t, x}(s), Z_2^{t, x}(s))$, $0 \leq s \leq T$, to (1.43) (with the same starting points of X , i.e. $(t_i, x_i) = (t, x)$, $i =$*

1, 2). Assume that $g_1(X^{t,x}(T)) \leq g_2(X^{t,x}(T))$ a.s. and $f_1(s, X^{t,x}(s), y, z) \leq f_2(s, X^{t,x}(s), y, z)$, $dt \times d\mathbb{P}$ a.e., $\forall (y, z) \in \mathbb{R} \times L_2^{\mathcal{Q}}(H; \mathbb{R}) (\cong \mathbb{R} \times H)$.

Then $Y_1^{t,x}(s) \leq Y_2^{t,x}(s)$, $t \leq s \leq T$, a.s.

Proof. By using Itô-Tanaka's formula ([52]) and taking expectation, we get

$$\begin{aligned} \mathbb{E} |(Y_1^{t,x} - Y_2^{t,x})^+(s)|^2 &= \mathbb{E} |(g_1(X^{t,x}(T)) - g_2(X^{t,x}(T)))^+|^2 \\ &+ 2 \mathbb{E} \int_s^T < (Y_1^{t,x}(r) - Y_2^{t,x}(r))^+, \\ &f_1(r, X^{t,x}(r), Y_1^{t,x}(r), Z_1^{t,x}(r)) - f_2(r, X^{t,x}(r), Y_2^{t,x}(r), Z_2^{t,x}(r)) > dr \\ &- \mathbb{E} \int_s^T 1_{\{Y_1^{t,x}(r) > Y_2^{t,x}(r)\}} |Z_1^{t,x}(r) - Z_2^{t,x}(r)|^2 dr \\ &- \frac{1}{2} \mathbb{E} (L^0(T) - L^0(s)), \end{aligned}$$

where $L^0(s)$ is the local time of the semimartingale $\{(Y_1^{t,x} - Y_2^{t,x})(s), s \in [0, T]\}$ at point 0, at time s , which is a continuous increasing process vanishing at $s = t$.

Thus, by using the assumptions in the theorem, (B2)" and (B3)", it follows that

$$\begin{aligned} \mathbb{E} |(Y_1^{t,x}(s) - Y_2^{t,x}(s))^+|^2 &\leq \\ 2(\mu + k) \mathbb{E} \int_s^T 1_{\{Y_1^{t,x}(r) > Y_2^{t,x}(r)\}} |Y_1^{t,x}(s) - Y_2^{t,x}(s)|^2 dr, \end{aligned}$$

which yields after using Gronwall's inequality that $(Y_1^{t,x}(s) - Y_2^{t,x}(s))^+ = 0$, $\forall s \in [t, T]$ a.s. ■

Remark 1.19 In the case the space H has finite dimension, the comparison in Theorem 1.18 is strict, in the sense that if, in addition to the assumption in Theorem 1.18, either $g_1(X^{t,x}(T)) < g_2(X^{t,x}(T))$ or $f_1(s, X^{t,x}(s), y, z) < f_2(s, X^{t,x}(s), y, z)$, $(s, y, z) \in [t, T] \times \mathbb{R} \times L_2^{\mathcal{Q}}(H; \mathbb{R})$ on a set of positive $dt \times d\mathbb{P}$ measure, then $Y_1^{t,x}(s) < Y_2^{t,x}(s)$, $t \leq s \leq T$ a.s.

This result can be found in [40, Theorem 1.6]. The idea of the proof comes from trying to show that the difference $(\delta Y(s), \delta Z(s)) := (Y_2^{t,x}(s) - Y_1^{t,x}(s), Z_2^{t,x}(s) - Z_1^{t,x}(s))$, $t \leq s \leq T$, actually solves a linear BSDE, namely,

$$\begin{aligned} \delta Y(s) = & \int_s^T [\gamma(r) \delta Y(r) + \langle \beta(r), \delta Z(r) \rangle_{L_2^{\mathcal{Q}}(H; \mathbb{R})}] dr \\ & - \int_s^T \delta_2 f(r) dr - \int_s^T \delta Z(r) dW(r), \end{aligned} \quad (1.50)$$

with γ and β that are progressively measurable and bounded processes. Using this new formulation one can find out how the solution to (1.50) looks like. Then from the above positivity assumption it becomes clear to see that $Y_2^{t,x}(t) - Y_1^{t,x}(t)$ is strictly positive.

Unfortunately, this method does not work, in general, when H has infinite dimension since it is not obvious how one can find such a process β , which we require to be bounded in order for the linear BSDE of type (1.50) to have a unique solution.

This problem arises from being W , the driving Wiener process, having infinite rank in H . Thus the proof of the above result can be extended to our case when W is finite dimensional, independent of H having a finite dimension.

Let \mathcal{L} denotes the following time dependent second-order differential operator acting on mappings $\Psi : [0, T] \times H \rightarrow K$, as follows

$$\begin{aligned} \mathcal{L}\Psi(t, x) : H &\rightarrow K, \\ \langle \mathcal{L}\Psi(t, x), e'_j \rangle_K &:= L\Psi_j, \quad j = 1, 2, \dots, \end{aligned} \quad (1.51)$$

where $\Psi_j \equiv \langle \Psi, e'_j \rangle_K$, $j = 1, 2, \dots$, and L acts on mappings $\psi : H \rightarrow \mathbb{R}$ as follows

$$\begin{aligned} L\psi(t, x) = & \frac{1}{2} \text{tr}_H [D^2\psi(t, x)(\sigma(t, x)\mathcal{Q}^{1/2})(\sigma(t, x)\mathcal{Q}^{1/2})^*] \\ & + \langle b(t, x), \nabla\psi(t, x) \rangle_H, \end{aligned}$$

is the infinitesimal generator of the Markov process $\{X^{t,x}(s), t \leq s \leq T\}$.

Consider the following second-order semilinear parabolic PDE

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), Du(t, x) \sigma(t, x)) = 0, \\ u(T, x) = g(x), \end{cases} \quad (1.52)$$

where $(t, x) \in [0, T] \times H$. This PDE must be understood in the following sense: for all $(t, x) \in [0, T] \times H$ and $j = 1, 2, \dots$,

$$\begin{cases} \frac{\partial}{\partial t} u_j(t, x) + Lu_j(t, x) + f_j(t, x, u(t, x), Du(t, x) \sigma(t, x)) = 0, \\ u(T, x) = g(x). \end{cases} \quad (1.53)$$

Here $u_i \equiv \langle u, e'_i \rangle_K$ and $f_j \equiv \langle f, e'_j \rangle_K$, $j = 1, 2, \dots$.

Write (1.41) as the following system of BSDEs:

$$\begin{aligned} Y_j^{t,x}(s) &= g_j(X^{t,x}(T)) + \int_s^T f_j(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r)) dr \\ &\quad - \int_s^T Z_j^{t,x}(r) dW(r), \end{aligned} \quad (1.54)$$

where Z_j is defined such that $Z_j(v) \equiv \langle Z(v), e'_j \rangle_K$, $j = 1, 2, \dots$, $v \in H$.

Theorem 1.20 *Assume that (1.41) has a unique solution, $(Y^{t,x}, Z^{t,x})$, and assume that (1.52) has a classical solution $u \in C^{1,2}([0, T] \times H; K)$. Then*

$$u(t, x) = Y^{t,x}(t), \quad (t, x) \in [0, T] \times H.$$

Proof. Let $y(s) := u_j(s, X^{t,x}(s))$. Then Itô's formula implies that

$$\begin{aligned} y(s) &= g_j(X^{t,x}(T)) - \int_s^T \left(\frac{\partial u_j}{\partial r}(r, X^{t,x}(r)) + Lu_j(r, X^{t,x}(r)) \right) dr \\ &\quad - \int_s^T Du_j(r, X^{t,x}(r))(\sigma(s, X^{t,x}(r))) dW(r), \\ &= \int_s^T f_j(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r)) dr \\ &\quad - \int_s^T Du_j(r, X^{t,x}(r))(\sigma(s, X^{t,x}(r))) dW(r), \end{aligned} \quad (1.55)$$

$t \leq s \leq T$. Write $z(s) := Du_j(s, X^{t,x}(s)) \sigma(s, X^{t,x}(s))$. By using the uniqueness property of the solution of (1.54) we conclude that $(Y_j, Z_j) = (y, z)$ a.s. In particular, a.s. $Y_j^{t,x}(s) = u_j(s, X^{t,x}(s))$ and $Z_j^{t,x}(s) = Du_j(s, X^{t,x}(s)) \sigma(s, X^{t,x}(s))$, for all $j = 1, 2, \dots$, and $s \in [0, T]$.

Since j is arbitrary, we have a.s. the following $Y^{t,x}(s) = u(s, X^{t,x}(s))$ and $Z^{t,x}(s) = Du(s, X^{t,x}(s)) \sigma(s, X^{t,x}(s))$, $t \leq s \leq T$. In particular, $u(t, x) = Y^{t,x}(t)$, since $Y^{t,x}(t)$ is deterministic. ■

Note that in this proof one can also use Itô's formula directly for $u(s, X^{t,x}(s))$, without going through its j^{th} components, and work with equations (1.52) and (1.41) directly.

This theorem generalises the well-known *Feynman-Kac formula*, e.g. when $f \equiv 0$, $u(t, x) = \mathbb{E} Y^{t,x}(T) = \mathbb{E} g(X^{t,x}(T))$.

Clearly, for such a mapping $u(t, x) \triangleq Y^{t,x}(t)$, $(t, x) \in [0, T] \times H$ to solve (1.52) we need to add some conditions on the mappings f, g and on the coefficients of \mathcal{L} , which make them regular enough to guarantee its ability to be differentiable. However, the following lemma proves the continuity of the mapping $[0, T] \times H \ni (t, x) \mapsto Y^{t,x}(t) \in K$.

Lemma 1.21 *The mapping u defined by $u(t, x) \triangleq Y^{t,x}(t)$, $(t, x) \in [0, T] \times H$, is continuous.*

Proof. Let $\{(t_n, x_n), n \geq 1\}$ be an arbitrary sequence in $[0, T] \times H$, such that $(t_n, x_n) \rightarrow (t, x)$ in $[0, T] \times H$ as $n \rightarrow \infty$.

It is sufficient to prove that $|u(t_n, x_n) - u(t, x)|_K \rightarrow 0$, as $n \rightarrow \infty$.

Since $(t, x) \mapsto X^{t,x}(T)$ is continuous in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$, then

$$\mathbb{E} |X^{t_n, x_n}(T) - X^{t, x}(T)|_H^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus there exists a subsequence $\{n_j\}_{j \geq 1}$ such that

$$X^{t_{n_j}, x_{n_j}}(T) \rightarrow X^{t, x}(T) \text{ a.s. , as } j \rightarrow \infty.$$

Moreover, since g is continuous, then

$$g(X^{t_{n_j}, x_{n_j}}(T)) \rightarrow g(X^{t,x}(T)) \text{ a.s. , as } j \rightarrow \infty .$$

This together with the uniform integrability of $\{g(X^{t_{n_j}, x_{n_j}}(T)) , j \geq 1\}$, taking values in K , which comes from (B1)” and the uniform integrability¹ of $\{X^{t_{n_j}, x_{n_j}}(T) , j \geq 1\}$, implies that

$$\mathbb{E} |g(X^{t_{n_j}, x_{n_j}}(T)) - g(X^{t,x}(T))|_K^2 \rightarrow 0, \text{ as } j \rightarrow \infty .$$

From this and Proposition 1.17 we get that

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |Y^{t_{n_j}, x_{n_j}}(s) - Y^{t,x}(s)|_K^2 \\ & \leq C (\mathbb{E} |g(X^{t_{n_j}, x_{n_j}}(T)) - g(X^{t,x}(T))|_K^2 \\ & + \mathbb{E} \int_0^T |f(r, X^{t_{n_j}, x_{n_j}}(r), Y^{t,x}(r), Z^{t,x}(r)) - \\ & f(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))|_K^2 dr) \\ & \rightarrow 0, \text{ as } j \rightarrow \infty . \end{aligned}$$

We have used for this result (B3)” and (B4)” and the uniform integrability of $\{f(X^{t_{n_j}, x_{n_j}}(r)) , j \geq 1\}$ (same argument as above).

But, by knowing that $Y^{t,x}(t)$ is deterministic since it is \mathcal{F}_t^t -measurable, we find that

$$\begin{aligned} |u(t_{n_j}, x_{n_j}) - u(t, x)|_K & \leq (\mathbb{E} \sup_{0 \leq s \leq T} |Y^{t,x}(s) - Y^{t_{n_j}, x_{n_j}}(s)|_K^2)^{1/2} \\ & \rightarrow 0 \text{ as } j \rightarrow \infty . \end{aligned}$$

This implies the continuity of u at (t, x) . The result then follows. ■

In the same method, one can also prove that the mapping $[0, T]^2 \times H \ni (t, s, x) \mapsto Y^{t,x}(s)$, taking value in K , is also continuous into L^2 with respect

¹E .g. when $p = 1$, one can use for this (1.42); otherwise it is just a simple matter of working out $\sup_{j \geq 1} \mathbb{E} |X^{t_{n_j}, x_{n_j}}(T)|_H^p$.

to the three variable. Furthermore, if the above mappings f and g are Lipschitz with respect to x , an inequality similar to (1.42) can be proved as well.

Denote respectively by $C_b([0, T] \times H)$ and $C_b^{1,2}([0, T] \times H)$ the space of bounded continuous functions from $[0, T] \times H$ into \mathbb{R} , and the space of real-valued functions on $[0, T] \times H$, which are Frechét differentiable, once in the first variable and twice in the second variable, and are continuous and bounded along with all their partial derivatives.

Let us now introduce the definition of *viscosity solutions* of the system of PDEs (1.52), following [20] and Lions [31] (see also his related later work).

Definition 1.22 (i) A mapping $u \in C([0, T] \times H; K)$ is called a viscosity subsolution of (1.52) if $u_j(T, x) \leq g_j(x)$, $x \in H$, $\forall j \geq 1$, and for any $j \geq 1$, $\phi \in C_b^{1,2}([0, T] \times H)$ the following holds for each local maximum point of $u_j - \phi$, $(t, x) \in [0, T) \times H$:

$$-\frac{\partial}{\partial t} \phi(t, x) - L\phi(t, x) - f_j(t, x, u(t, x), D\phi(t, x) \sigma(t, x)) \leq 0.$$

(ii) A mapping $u \in C([0, T] \times H; K)$ is called a viscosity supersolution of (1.52) if $u_j(T, x) \geq g_j(x)$, $x \in H$, $\forall j \geq 1$, and for any $j \geq 1$, $\phi \in C_b^{1,2}([0, T] \times H)$ the following holds for each local minimum point of $u_j - \phi$, $(t, x) \in [0, T) \times H$:

$$-\frac{\partial}{\partial t} \phi(t, x) - L\phi(t, x) - f_j(t, x, u(t, x), D\phi(t, x) \sigma(t, x)) \geq 0.$$

(iii) A mapping $u \in C([0, T] \times H; K)$ is called a viscosity solution of (1.52) if u is both a viscosity sub- and super solution of (1.52).

To make use of this definition for the above system of PDEs (1.52), we have to force f_j to depend only on Z_j for each $j \geq 1$.

Theorem 1.23 Suppose that $\dim H < \infty$. Under $(B1)''$ – $(B4)''$, $u(t, x) \triangleq Y^{t,x}(t)$, $(t, x) \in [0, T] \times H$ is a viscosity solution of (1.52).

Proof. We can assume without loss of generality that W is a d -dimensional Brownian motion in \mathbb{R}^d .

We shall only prove that u is a viscosity subsolution of (1.52), as it can be seen in an analogous way that u is a viscosity supersolution of (1.52). We have already shown in Lemma 1.21 that such u is continuous.

Take any $j \geq 1$. Let $\phi \in C_b^{1,2}([0, T] \times H)$ and let $(t, x) \in [0, T] \times H$ be a local maximum point of $u_j - \phi$. We assume without loss of generality that

$$u_j(t, x) = \phi(t, x). \quad (1.56)$$

Suppose that

$$-\frac{\partial}{\partial t} \phi(t, x) - L\phi(t, x) - f_j(t, x, u(t, x), D\phi(t, x) \sigma(t, x)) > 0. \quad (1.57)$$

It follows then that there exists $\alpha \in (0, T - t)$ such that $u_j(s, y) \leq \phi(s, y)$ and (1.57) holds for every $(s, y) \in [t, t + \alpha] \times B_\alpha(x)$, where $B_\alpha(x) := \{x \in H, |x - y|_H \leq \alpha\}$.

Define the stopping time $\tau := \inf\{s > t : X^{t,x}(s) \notin B_\alpha(x)\} \wedge (t + \alpha)$. Then $(\bar{Y}(s), \bar{Z}(s)) := (Y_j^{t,x}(s \wedge \tau), 1_{[t, \tau]}(s) Z_j^{t,x}(s))$, $t \leq s \leq t + \alpha$, solves the following BSDE

$$\begin{aligned} \bar{Y}(s) &= u_j(\tau, X^{t,x}(\tau)) + \int_{\tau \wedge s}^{\tau} f_j(r, X^{t,x}(r), u(r, X^{t,x}(r)), \bar{Z}(r)) dr \\ &\quad - \int_s^{t+\alpha} \bar{Z}(r) dW(r). \end{aligned}$$

Note that Itô's formula implies that $(\hat{Y}(s), \hat{Z}(s)) := (\phi(s \wedge \tau, X^{t,x}(s)), 1_{[t, \tau]}(s) D\phi(s, X^{t,x}(s)) \sigma(s, X^{t,x}(s)))$, $t \leq s \leq t + \alpha$, solves the following BSDE

$$\begin{aligned} \hat{Y}(s) &= \phi(\tau, X^{t,x}(\tau)) - \int_{\tau \wedge s}^{\tau} \left[\frac{\partial}{\partial r} \phi(r, X^{t,x}(r)) + L\phi(r, X^{t,x}(r)) \right] dr \\ &\quad - \int_s^{t+\alpha} \hat{Z}(r) dW(r). \end{aligned}$$

Finally, from the comparison theorem, see also Remark 1.19, we conclude that $\bar{Y}(t) < \hat{Y}(t)$, i.e. $u_j(t, x) < \phi(t, x)$. But this contradicts (1.56) . ■

In the following theorem we let W be an infinite dimensional Wiener process. We then make the following restriction on the PDE (1.52), consequently the same is made to (1.41) as well. We then have

$$\begin{aligned} Y^{t,x}(s) = & g(X^{t,x}(T)) + \int_s^T f(r, X^{t,x}(r), Y^{t,x}(r)) dr \\ & - \int_s^T Z^{t,x}(r) dW(r), \end{aligned} \quad (1.58)$$

in which the mapping f drops its depending on Z . Similarly, f in the following system of Parabolic PDEs drops its dependence on the derivative of u , in particular, we are in charge of

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x)) = 0, \\ u(T, x) = g(x). \end{cases} \quad (1.59)$$

Theorem 1.24 *Under $(B1)'' - (B4)''$, $u(t, x) \triangleq Y^{t,x}(t)$, $(t, x) \in [0, T] \times H$, is a viscosity solution of (1.59), where $Y^{t,x}(s)$, $t \leq s \leq T$, is the unique solution of (1.58).*

Proof. The continuity of u was proved in Lemma 1.21.

Take any $j \geq 1$, $\phi \in C_b^{1,2}([0, T] \times H)$, and $(t, x) \in [0, T] \times H$ such that $u_j(t, x) = \phi(t, x)$ and $u_j \leq \phi$ otherwise. Suppose for a contrary that the following is true:

$$-\frac{\partial}{\partial t} \phi(t, x) - L\phi(t, x) - f(t, x, u(t, x)) > 0. \quad (1.60)$$

As in the previous proof, there exists $\alpha \in (0, T - t)$ such that (1.60) holds, for every $(s, y) \in [t, t + \alpha] \times B_\alpha(x)$.

We use here the same definition as in the preceding proof for τ and define $(\bar{Y}(s), \bar{Z}(s)) := (Y_j^{t,x}(s \wedge \tau), 1_{[t, \tau]}(s) Z_j^{t,x}(s))$, $t \leq s \leq t + \alpha$. It is the solution

of

$$\begin{aligned}\bar{Y}(s) = & u_j(\tau, X^{t,x}(\tau)) + \int_{\tau \wedge s}^{\tau} f_j(r, X^{t,x}(r), u(r, X^{t,x}(r))) dr \\ & - \int_s^{t+\alpha} \bar{Z}(r) dW(r).\end{aligned}$$

For $t \leq s \leq t + \alpha$, denote by $(\hat{Y}(s), \hat{Z}(s)) := (\phi(s \wedge \tau, X^{t,x}(s)), 1_{[t,\tau]}(s) D\phi(s, X^{t,x}(s)) \sigma(s, X^{t,x}(s)))$, which then solves the following BSDE

$$\begin{aligned}\hat{Y}(s) = & \phi(\tau, X^{t,x}(\tau)) - \int_{\tau \wedge s}^{\tau} \left[\frac{\partial}{\partial r} \phi(r, X^{t,x}(r)) + L\phi(r, X^{t,x}(r)) \right] dr \\ & - \int_s^{t+\alpha} \hat{Z}(r) dW(r).\end{aligned}$$

It is immediately seen from the comparison theorem (Theorem 1.18) that $\bar{Y}(t) \leq \hat{Y}(t)$, i.e. $u_j(t, x) \leq \phi(t, x)$. We now show that $u_j(t, x) < \phi(t, x)$ contradicting the earlier assumption $u_j(t, x) = \phi(t, x)$. Note that

$$\begin{aligned}\bar{Y}(t) - \hat{Y}(t) = & \mathbb{E} [u_j(\tau, X^{t,x}(\tau)) - \phi(\tau, X^{t,x}(\tau)) \\ & + \int_t^{\tau} \left(\frac{\partial}{\partial r} \phi(r, X^{t,x}(r)) + L\phi(r, X^{t,x}(r)) + f_j(r, X^{t,x}(r), u(r, X^{t,x}(r))) \right) dr].\end{aligned}$$

Thus $\bar{Y}(t) < \hat{Y}(t)$, which gives that $u_j(t, x) < \phi(t, x)$. Therefore u is a viscosity subsolution of (1.59).

The proof that u is a viscosity supersolution of (1.59) is done analogously. The proof is then complete. ■

Theorem(s) (1.23 , 1.24) prove only existence of viscosity solutions to the system of parabolic semilinear PDEs, (1.52) and (1.59) respectively. The uniqueness case remains an interesting research problem.

Chapter 2

Backward Stochastic Evolution Equations in Infinite Dimensions

2.1 Backward Stochastic Evolution Equations

Let A be a second order operator on K , possibly unbounded, which generates a C_0 -semigroup, $\{e^{At}, t \geq 0\}$, on K . From here on we will assume that W is cylindrical Wiener process on H together with its Wiener filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

Consider the following BSEE

$$\begin{cases} -dY(t) = A Y(t) dt + f(t, Y(t), Z(t)) dt - Z(t) dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi. \end{cases} \quad (2.1)$$

We say that the pair $(Y, Z) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$ is a solution to (2.1) if it satisfies the following equality

$$\begin{aligned} Y(t) = e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds \\ - \int_t^T e^{A(s-t)} Z(s) dW(s), \quad 0 \leq t \leq T. \end{aligned} \quad (2.2)$$

Thus our solutions here, in this sense, are all *mild* solutions.

Such equations are useful in studying stochastic Hamilton-Jacobi-Bellman equations, as gleaned from the work in [48].

Let us now set some assumptions which guarantee the existence and uniqueness of such a solution.

- (C1) f is a mapping from $[0, T] \times \Omega \times K \times L_2(H; K)$ to H that is $\mathcal{P} \otimes \mathcal{B}(K) \otimes \mathcal{B}(L_2(H; K)) / \mathcal{B}(K)$ -measurable. We assume that f satisfy the following integrability condition:

$$f(\cdot, 0, 0) \in L^2_{\mathcal{F}}(0, T; K),$$

- (C2) $\exists k > 0$ such that $\forall y, y' \in K$ and $\forall z, z' \in L_2(H; K)$

$$|f(t, y, z) - f(t, y', z')|_K^2 \leq c(|y - y'|_K^2) + k |z - z'|_{L_2(H; K)}^2,$$

for a.e. $(\omega, t) \in \Omega \times [0, T]$, where c is a concave nondecreasing continuous function from \mathbb{R}_+ to \mathbb{R}_+ such that $c(0) = 0, c(x) > 0$ for any $x > 0$ and

$$\int_{0+}^a \frac{dx}{c(x)} = \infty,$$

for any sufficiently small $a > 0$.

An example of such a function c is $c(x) = \alpha x$, $\alpha > 0$. Thus our result generalises the result of [25]. The following examples were taken from [33]. Let $\delta \in (0, 1)$ be sufficiently small and define

$$c_1(x) = \begin{cases} x \log(x^{-1}) & \text{if } 0 \leq x \leq \delta, \\ \delta \log(\delta^{-1}) + \dot{c}_1(\delta-) (x - \delta) & \text{if } x > \delta; \end{cases}$$

$$c_2(x) = \begin{cases} x \log(x^{-1}) \log \log(x^{-1}) & \text{if } 0 \leq x \leq \delta, \\ \delta \log(\delta^{-1}) \log \log(\frac{1}{\delta}) + \dot{c}_2(\delta-) (x - \delta) & \text{if } x > \delta. \end{cases}$$

Then $c := c_i$, $i = 1, 2$, satisfies the above conditions in (C2). Another example can be found in [9], where the above assumptions on c are clearly checked.

Let us record here that introducing such a condition in (C2) in the study of uniqueness of solutions of stochastic differential equations is due to Yamada and Watanabe in [59] and [60].

Our main theorem of this section is the following.

Theorem 2.1 *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$ be given and f satisfy the above assumptions, in particular (C1), (C2). Then there exists a unique pair $(Y, Z) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$ which solves (2.1) .*

N.B. in Section 4 below we shall be dealing with a more general BSEE than (2.1) and, in particular, we will let the operator A be a time dependent closed operator, i.e. $A(t)$, for $0 \leq t \leq T$. In which case the semigroup $\{e^{A(s-t)}, 0 \leq t \leq s\}$ must be replaced by a set of bounded linear operators, which we denote by $\{U(s, t), 0 \leq t \leq s\}$. This U will be taken as a strong evolution operator in the sense of Definition 2.20 below. However, we will not give a precise proof for this latter case since its proof stays a copy word by word of our proof of the semigroup case when A is not time dependent, once we replace $e^{A(s-t)}$ by $U(s, t)$. See, for example, the change of proof made to Lemma 2.23 below, which is a generalisation of the following lemma.

Before giving the proof of Theorem 2.1, let us introduce some lemmas which will help to establish it. The following lemma is a special case of the theorem when f and g in (2.2) do not depend on Y and Z .

Lemma 2.2 *If $f \in L^2_{\mathcal{F}}(0, T; K)$ and $\xi \in L^2(\Omega, \mathcal{F}_T; \mathbb{P}; K)$, then there exists a unique pair $(Y, Z) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$ such that*

$$Y(t) = e^{A(T-t)}\xi + \int_t^T e^{A(s-t)} f(s) ds - \int_t^T e^{A(s-t)} Z(s) dW(s). \quad (2.3)$$

Furthermore, $\forall t \in [0, T]$,

$$\mathbb{E} |Y(t)|_K^2 \leq 2M^2(T-t) \mathbb{E} \int_t^T |f(s)|_K^2 ds + 2M^2 \mathbb{E} |\xi|_K^2, \quad (2.4)$$

$$\mathbb{E} \int_t^T |Z(s)|_{L_2(H;K)}^2 ds \leq 8 M^2 (T-t) \mathbb{E} \int_t^T |f(s)|_K^2 ds + 8 M^2 \mathbb{E} |\xi|_K^2 ds, \quad (2.5)$$

where

$$M := \sup_{t \in [0, T]} \{ \| e^{At} \| \}.$$

The proof of this lemma can be found in [25, Lemma 2.1]. We will sketch a few details for the sake of completeness, and also for the reader to see how the solution of (2.3) looks like.

Proof. (Sketch)

Uniqueness : Let both (Y_1, Z_1) and (Y_2, Z_2) be two solutions of (2.3). Then for arbitrary $t \in [0, T]$

$$Y_1(t) - Y_2(t) = \int_t^T e^{A(s-t)} (Z_1(s) - Z_2(s)) dW(s). \quad (2.6)$$

By applying conditional expectation $\mathbb{E} [\cdot | \mathcal{F}_t]$ to both sides of (2.6) and using the continuity of Y_1 and Y_2 , cf. Proposition 2.10 below, we obtain

$$Y_1(t) = Y_2(t), \quad \forall t \in [0, T] \text{ a.s.}$$

Hence, by a simple use of (1.5), we find that $Z_1(t) = Z_2(t)$, for all $t \in [0, T]$, a.s.

Existence : Define

$$Y(t) = \mathbb{E} [e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} f(s) ds | \mathcal{F}_t], \quad 0 \leq t \leq T. \quad (2.7)$$

Hence (2.4) follows immediately from Jensen's inequality and assumption (C1).

To construct $Z(\cdot)$, we use the martingale representation theorem (Theorem 1.2) as follows. Since for each $s \in [0, T]$, $f(s)$ and ξ belong to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$, there exist two process $z_1(s)$ and z_2 in $L^2_{\mathcal{F}}(0, T; L_2(H; K))$, such that

$$\mathbb{E} [f(s) | \mathcal{F}_t] = \mathbb{E} f(s) + \int_0^t z_1(s)(r) dW(r), \quad 0 \leq t \leq s, \quad (2.8)$$

and

$$\mathbb{E} [\xi | \mathcal{F}_t] = \mathbb{E} \xi + \int_0^t z_2(r) dW(r), \quad 0 \leq t \leq T. \quad (2.9)$$

It is not difficult to see that $z_1(\cdot)(\cdot)$ is $\mathcal{B}([0, T]) \otimes \mathcal{P}$ -measurable.

Now take

$$Z(t) := e^{A(T-t)} z_2(t) + \int_t^T e^{A(s-t)} z_1(s)(t) ds, \quad (2.10)$$

for all $0 \leq t \leq T$. It is then easy to check that (2.3) holds. The estimate (2.5) follows from (2.10), (2.8) and (2.9). ■

Example 2.3 *Coping with the above setting, let $K = L^2(\mathbb{R}^n; \mathbb{R})$ and $A = \frac{1}{2} \Delta$. Consider the following BSEE*

$$\begin{aligned} -dY(t) &= \frac{1}{2} \Delta Y(t) dt + Z(t) dW(t), \\ Y(T) &= \Phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(\mathbb{R}^n, \mathbb{R})). \end{aligned}$$

In this case,

$$Y(t, x) = e^{-\frac{1}{2} \Delta(T-t)} \mathbb{E} \Phi(t, x) - \int_0^t e^{-\frac{1}{2} \Delta(T-t)} z_2(s)(x) dW(s)$$

and

$$Z(t, x) = (e^{-\frac{1}{2} \Delta(T-t)} z_2(t))(x),$$

$t \in [0, T]$, $x \in \mathbb{R}^n$. The process z_2 is given as in the lemma through the martingale representation theorem, hence z_2 can be calculated explicitly if Φ is regular enough according to the Clark-Ocone theorem, cf. Theorem 1.8.

Remark 2.4 *For $p > 2$, using (2.7) we have also sort of moment inequalities for the solution of BSEE (2.3)*

$$\mathbb{E} |Y(t)|_K^p \leq 2^{p-1} M^p \mathbb{E} |\xi|_K^p + 2^{p-1} M^p (T-t)^{p-1} \mathbb{E} \int_t^T |f(s)|_K^p ds,$$

which yields

$$\begin{aligned} \mathbb{E} \left| \int_t^T e^{A(s-t)} Z(s) dW(s) \right|_K^p &\leq 3^{p-1} M^p (2^{p-1} + 1) \mathbb{E} |\xi|_K^p \\ &+ 3^{p-1} M^p (T-t)^{p-1} (2^{p-1} + 1) \mathbb{E} \int_t^T |f(s)|_K^p ds, \end{aligned}$$

for all $t \in [0, T]$.

These inequalities become useful when the right hand side of both of them is finite.

Proposition 2.5 *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$ and $f : \Omega \times [0, T] \times L_2(H; K) \rightarrow K$ be a mapping satisfying (C1) and (C2). Then there exists a unique solution $(Y, Z) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$ to the following BSEE*

$$\begin{cases} -dY(t) = A Y(t) dt + f(t, Z(t)) dt - Z(t) dW(t), \\ Y(T) = \xi. \end{cases} \quad (2.11)$$

The proof of this proposition can be found in [25, Proposition 2.1].

We now study the BSEE (2.1). Let us first introduce, with the help of Proposition 2.5, the following iteration scheme, from which, we will be able to construct the solution of (2.1). Let $Y_0(t) \equiv 0$ and let $\{(Y_n(t), Z_n(t)) : 0 \leq t \leq T, n \geq 1\}$ be a sequence in $L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$, defined recursively as follows:

$$\begin{aligned} Y_n(t) = e^{A(T-t)} \xi &+ \int_t^T e^{A(s-t)} f(s, Y_{n-1}(s), Z_n(s)) ds - \\ &\int_t^T e^{A(s-t)} Z_n(s) dW(s), \quad 0 \leq t \leq T. \end{aligned} \quad (2.12)$$

We shall show, through couple of lemmas, that these solutions $\{(Y_n, Z_n)\}$ to (2.12) actually converge to the solution of the original BSEE (2.1) as $n \rightarrow \infty$.

Lemma 2.6 Assume that hypotheses (C1) and (C2) are imposed. Then there exist positive constants C_1 and C_2 such that for all $t \in [0, T]$ and $n \geq 1$, the solution of (2.12) satisfies the following

$$\mathbb{E} |Y_n(t)|_K^2 \leq C_1, \text{ and } \mathbb{E} \int_0^T |Z_n(s)|_{L_2(H;K)}^2 ds \leq C_2.$$

Proof. By applying Lemma 2.2 to equation (2.12) we see that

$$\mathbb{E} |Y_n(t)|_K^2 \leq 2M^2(T-t) \mathbb{E} \int_t^T |f(s, Y_{n-1}(s), Z_n(s))|_K^2 ds + 2M^2 \mathbb{E} |\xi|_K^2 \quad (2.13)$$

and

$$\begin{aligned} \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds &\leq 8M^2(T-t) \mathbb{E} \int_t^T |f(s, Y_{n-1}(s), Z_n(s))|_K^2 ds \\ &\quad + 8M^2 \mathbb{E} |\xi|_K^2. \end{aligned} \quad (2.14)$$

Moreover, by using (C2), we find that

$$\begin{aligned} \mathbb{E} \int_t^T |f(s, Y_{n-1}(s), Z_n(s))|_K^2 ds &\leq \\ C_3 + 2b \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds + 2k \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds, \end{aligned} \quad (2.15)$$

since c is concave and so there exists $a, b > 0$, such that $c(x) \leq a + bx$; here $C_3 := 2aT + 2 \mathbb{E} \int_0^T |f(s, 0, 0)|^2 ds$. Thus (2.13) and (2.14) take the following shape

$$\begin{aligned} \mathbb{E} |Y_n(t)|_K^2 &\leq 2M^2(T-t)C_3 + 4M^2(T-t)b \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds \\ &\quad + 4M^2(T-t)k \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds + 2M^2 \mathbb{E} |\xi|_K^2 \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds &\leq \\ 8M^2 \mathbb{E} |\xi|_K^2 + 8M^2(T-t)C_3 + 16M^2(T-t)b \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds \\ + 16M^2(T-t)k \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds. \end{aligned} \quad (2.17)$$

Take $\eta \in (0, T)$ with $32M^2k\eta < 1$, and assume that $t \in [T - \eta, T]$. Then we conclude that

$$\begin{aligned} \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds &\leq \\ &8M^2 \mathbb{E} |\xi|_K^2 + 8M^2\eta C_3 + 16M^2 \eta b \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds \\ &+ \frac{1}{2} \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds, \end{aligned} \quad (2.18)$$

or in particular,

$$\mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds \leq \frac{C_3}{2k} + 16M^2 \mathbb{E} |\xi|_K^2 + \frac{b}{k} \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds. \quad (2.19)$$

Similarly from (2.16) we get

$$\begin{aligned} \mathbb{E} |Y_n(t)|_K^2 &\leq \frac{C_3}{16k} + \frac{b}{8k} \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds \\ &+ \frac{1}{8} \mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds + 2M^2 \mathbb{E} |\xi|_K^2, \end{aligned} \quad (2.20)$$

from which and from (2.19), we obtain

$$\mathbb{E} |Y_n(t)|_K^2 \leq C_4 + C_5 \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds, \quad (2.21)$$

$\forall n \geq 1$, where the constants C_4, C_5 are $\frac{C_3}{8k} + 4M^2 \mathbb{E} |\xi|_K^2$ and $\frac{b}{4k}$ respectively.

In particular, (2.21) holds for all $n \geq 1$.

Fix an integer $m \geq 1$. Then if $1 \leq n \leq m$, we have

$$\mathbb{E} |Y_n(t)|_K^2 \leq C_4 + C_5 \mathbb{E} \int_t^T \sup_{1 \leq q \leq m} \mathbb{E} |Y_q(s)|_K^2 ds.$$

Thus, by Gronwall's inequality, we deduce that

$$\begin{aligned} \sup_{1 \leq q \leq m} \mathbb{E} |Y_q(t)|_K^2 &\leq C_4 e^{C_5(T-t)} \\ &< C_4 e^{C_5 T}. \end{aligned}$$

Since m is arbitrary, we get that

$$\mathbb{E} |Y_n(t)|_K^2 \leq C_6, \quad (2.22)$$

$\forall n \geq 1$ and $\forall t \in [T - \eta, T]$, where $C_6 := C_4 e^{C_5 T}$.

On the other hand, we obtain the corresponding estimate for Z . Let us first re-write (2.19) as in the following form

$$\mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds \leq C_7 + \frac{1}{k} \mathbb{E} \int_t^T |Y_{n-1}(s)|_K^2 ds, \quad (2.23)$$

where $C_7 := \frac{C_3}{2k} + 16M^2 \mathbb{E} |\xi|_K^2$. Then it follows by using (2.22) that

$$\mathbb{E} \int_t^T |Z_n(s)|_{L_2(H;K)}^2 ds \leq C_7 + \frac{b}{k} C_6 T =: C_8. \quad (2.24)$$

Next we assume that $t \in [T - 2\eta, T - \eta]$ and η is, as before, satisfying $32M^2 k \eta < 1$. Note that since

$$\begin{aligned} Y_n(t) &= Y_n(T - \eta) + \int_t^{T-\eta} f(s, Y_n(s), Z_n(s)) ds \\ &\quad - \int_t^{T-\eta} Z_n(s) dW(s), \end{aligned}$$

we can still use the same way in which we derived the inequalities (2.19) and (2.20) with slightly minor changes to derive eventually the following ones:
 $\forall t \in [T - 2\eta, T - \eta]$,

$$\begin{aligned} \mathbb{E} |Y_n(t)|_K^2 &\leq \frac{b}{8k} \mathbb{E} \int_t^{T-\eta} |Y_{n-1}(s)|_K^2 ds + 2M^2 \mathbb{E} |Y_n(T - \eta)|_K^2 \\ &\quad + \frac{C_3}{16k} + \frac{1}{8} \mathbb{E} \int_t^{T-\eta} |Z_n(s)|_{L_2(H;K)}^2 ds, \end{aligned} \quad (2.25)$$

$$\begin{aligned} \mathbb{E} \int_t^{T-\eta} |Z_n(s)|_{L_2(H;K)}^2 ds &\leq \frac{C_3}{2k} + \frac{b}{k} \mathbb{E} \int_t^{T-\eta} |Y_{n-1}(s)|_K^2 ds \\ &\quad + 16M^2 \mathbb{E} |Y_n(T - \eta)|_K^2. \end{aligned} \quad (2.26)$$

Therefore, by substituting (2.26) in (2.25) and using (2.22), we obtain

$$\mathbb{E} |Y_n(t)|_K^2 \leq C'_4 + C_5 \mathbb{E} \int_t^{T-\eta} |Y_{n-1}(s)|_K^2 ds, \quad (2.27)$$

by using (2.26) and (2.22), where $C'_4 := \frac{C_3}{8k} + 4M^2C_6$. By doing the same procedure as done for (2.22), we find that

$$\mathbb{E} |Y_n(t)|_K^2 \leq C'_4 e^{C_5(T-\eta-t)}, \quad \forall n \geq 1.$$

In particular,

$$\mathbb{E} |Y_n(t)|_K^2 \leq C'_6, \quad (2.28)$$

for all $t \in [t - 2\eta, T - \eta]$, where $C'_6 := C'_4 e^{C_5 T}$.

On the other hand, (2.28) implies that

$$\begin{aligned} \mathbb{E} \int_t^{T-\eta} |Z_n(s)|_{L_2(H;K)}^2 ds &\leq \frac{C_3}{2k} + 16M^2C_6 + \frac{b}{k} C'_6 (T - \eta - t) \\ &< C'_8, \end{aligned} \quad (2.29)$$

$\forall t \in [T - 2\eta, T - \eta]$ and $\forall n \geq 1$, where $C'_8 := \frac{C_3}{2k} + 16M^2C_6 + \frac{bT}{k} C'_6$.

Thus we have proved the two cases when t lies in either of the two intervals $[T - \eta, T]$ or $t \in [T - 2\eta, T - \eta]$. By repeating this procedure for every tiny interval, we derive similar inequalities to those (2.22), (2.24), (2.28) and (2.29). For example, recall that η was chosen to be small enough so that $32M^2k\eta < 1$, hence, if $t \in [0, T]$, then t should lie in one of the intervals $\{[(T - (l+1)\eta) \vee 0, T - l\eta], \quad 0 \leq l \leq q \text{ and } l \text{ is an integer}\}$, of length, at most, η , where q is the smallest integer such that $q \geq \frac{T}{\eta}$. Therefore, as done earlier for the two cases $l = 0$ and $l = 1$, we can easily obtain that

$$\mathbb{E} |Y_n(t)|_K^2 \leq C_9,$$

and

$$\mathbb{E} \int_t^{T-l\eta} |Z_n(s)|_{L_2(H;K)}^2 ds \leq C'_9,$$

$\forall n \geq 1$ and $t \in [(T - (l + 1)\eta) \vee 0, T - l\eta]$, for some positive constants C_9 and C'_9 .

The same also holds when varying l over $0 \leq l \leq q$. Thus, in particular, it follows (e.g. by summing on l over $0 \leq l \leq q$, if necessary) that $\forall t \in [0, T]$ and $\forall n \geq 1$, we have

$$\mathbb{E} |Y_n(t)|_K^2 \leq C_1$$

and

$$\mathbb{E} \int_0^T |Z_n(s)|_{L_2(H;K)}^2 ds \leq C_2,$$

for some constants $C_1 > 0$ and $C_2 > 0$. This prove the Lemma. ■

Lemma 2.7 *If hypotheses (C1), (C2) hold, then there exists constants $C_{10} > 0$ and $C'_{10} > 0$ such that $\forall 0 \leq t \leq T$ and $\forall n, m \geq 1$, we have*

$$\mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \leq C_{10} \int_t^T c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds, \quad (2.30)$$

and

$$\begin{aligned} & \mathbb{E} \int_t^T |Z_{n+m}(s) - Z_n(s)|_{L_2(H;K)}^2 ds \leq \\ & C'_{10} \int_t^T c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds. \end{aligned} \quad (2.31)$$

Proof. Note that

$$\begin{aligned} Y_{n+m}(t) - Y_n(t) = & \int_t^T e^{A(s-t)} [f(s, Y_{n+m-1}(s), Z_{n+m}(s)) - f(s, Y_{n-1}(s), Z_n(s))] ds \\ & - \int_t^T e^{A(s-t)} [Z_{n+m}(s) - Z_n(s)] dW(s). \end{aligned} \quad (2.32)$$

Thus, by using Lemma 2.2, we deduce the following two inequalities

$$\begin{aligned} & \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \leq \\ & M^2(T - t) \mathbb{E} \int_t^T c(|Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\ & + M^2(T - t) k \mathbb{E} \int_t^T |Z_{n+m}(s) - Z_n(s)|_{L_2(H;K)}^2 ds \end{aligned} \quad (2.33)$$

and

$$\begin{aligned}
\mathbb{E} \int_t^T |Z_{n+m}(s) - Z_n(s)|_{L_2(H;K)}^2 ds \leq \\
8 M^2 (T - t) \mathbb{E} \int_t^T c(|Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\
+ 8 M^2 (T - t) k \mathbb{E} \int_t^T |Z_{n+m}(s) - Z_n(s)|_{L_2(H;K)}^2 ds. \quad (2.34)
\end{aligned}$$

As done in the proof of Lemma 2.6, we divide the interval $[0, T]$ into tiny sub-intervals of length η such that $0 \leq \eta \leq T$ and $16M^2\eta k < 1$. We treat only the two cases individually: where $t \in [T - \eta, T]$ and where $t \in [T - 2\eta, T - \eta]$. The general case follows directly by recalling the argument stated at the end of the proof of the foregoing lemma.

Let us now study the case where $t \in [T - \eta, T]$. Then (2.34) becomes

$$\begin{aligned}
\mathbb{E} \int_t^T |Z_{n+m}(s) - Z_n(s)|_{L_2(H;K)}^2 ds \leq \\
\frac{1}{k} \mathbb{E} \int_t^T c(|Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds. \quad (2.35)
\end{aligned}$$

Therefore by (2.33) and (2.35), we have

$$\mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \leq C_{11} \int_t^T \mathbb{E} c(|Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds, \quad (2.36)$$

where $C_{11} := \frac{9}{16k}$. Thus by using *Jensen's inequality* (if $L : \mathbb{R} \rightarrow \mathbb{R}_+$ is a continuous concave function and X is a random variable in \mathbb{R} , then $\mathbb{E} L(X) \leq L(\mathbb{E} X)$), we deduce that

$$\mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \leq C_{11} \int_t^T c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds, \quad (2.37)$$

for all $t \in [T - \eta, T]$ and $n, m \geq 1$.

Next we assume that $t \in [T - 2\eta, T - \eta]$. Re-write (2.32) as the following

$$\begin{aligned}
Y_{n+m}(t) - Y_n(t) = & e^{A(T-\eta-t)}(Y_{n+m}(T-\eta) - Y_n(T-\eta)) \\
& + \int_t^{T-\eta} e^{A(s-t)} [f(s, Y_{n+m-1}(s), Z_{n+m}(s)) - f(s, Y_{n-1}(s), Z_n(s))] ds \\
& - \int_t^{T-\eta} e^{A(s-t)} [Z_{n+m}(s) - Z_n(s)] dW(s). \tag{2.38}
\end{aligned}$$

Apply Lemma 2.2 to equation (2.38) and use the assumption $(16M^2\eta k < 1)$, inequality (2.37) and Jensen's inequality to see that

$$\begin{aligned}
\mathbb{E} \int_t^{T-\eta} |Z_{n+m}(s) - Z_n(s)|_{L_2(H;K)}^2 ds & \leq \\
& \left(\frac{1}{2k}\right) \int_t^{T-\eta} c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\
& + 8M^2 C_{11} \int_{T-\eta}^T c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\
& \leq \left(\frac{1}{2k} + 8M^2 C_{11}\right) \int_t^T c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds, \tag{2.39}
\end{aligned}$$

$\forall t \in [T - 2\eta, T - \eta]$ and $\forall n, m \geq 1$.

Again by using Lemma 2.2 (or simply by taking conditional expectation on (2.38) and using Jensen's inequality) and using (2.37), (2.39) and Jensen's inequality, we deduce that

$$\mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \leq C_{12} \int_t^T c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds, \tag{2.40}$$

$\forall t \in [T - 2\eta, T - \eta]$ and $\forall n, m \geq 1$, where $C_{12} := \frac{3}{16k} + 3M^2 C_{11}$.

On the other hand, note that (2.35) and (2.39) prove (2.31) for the case where $t \in [T - 2\eta, T]$.

Finally, as introduced earlier in this proof, this is enough to derive (2.30) and (2.31) for any $t \in [0, T]$. The proof is complete. ■

Lemma 2.8 *Under hypotheses (C1) and (C2), there exists a constant $C_{13} > 0$ such that for all $t \in [0, T]$ and $n, m \geq 1$,*

$$\mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \leq C_{13} (T - t).$$

Proof. From Lemma 2.7 and Lemma 2.6 it follows that

$$\begin{aligned} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 &\leq C_{10} \int_t^T c(4C_1) ds \\ &= C_{13} (T - t), \end{aligned} \tag{2.41}$$

where $C_{13} := C_{10} c(4 C_1)$. ■

Let us now state *Bihari's inequality*, which we will need in our proof below; see [5, P. 83] or [33, Lemma 3.6] for the proof.

Proposition 2.9 (Bihari's inequality) *Let u be and v be two positive continuous functions on $[0, T]$ and $K \geq 0$, $C \geq 0$. Let $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonnegative nondecreasing continuous function. Then the inequality*

$$u(t) \leq K + C \int_0^t v(s) c(u(s)) ds, \quad 0 \leq t \leq T$$

implies

$$u(t) \leq G^{-1}(G(K) + C \int_0^t v(s) ds),$$

for all such $t \in [0, T]$ that $G(K) + \int_0^t v(s) ds \in \text{Dom} (G^{-1})$, where

$$G(r) := \int_1^r \frac{ds}{c(s)}, \quad r > 0$$

and G^{-1} is the inverse function of G .

In particular, if $K = 0$ and

$$\lim_{r \rightarrow 0^+} G(r) = -\infty,$$

then $u(t) = 0$, $0 \leq t \leq T$.

Proof of Theorem 2.1.

Existence : We argue as Mao did in the proof of [33, Theorem 2.1].

We claim first that, for any $m \geq 1$,

$$\sup_{t \in [0, T]} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (2.42)$$

Recall the two constants C_{10} and C_{13} from Lemma 2.7 and Lemma 2.8. Denote by \bar{c} the function $C_{10} c$, defined on \mathbb{R}_+ . Let $\zeta_1 \in [0, T]$ be such that $\bar{c}(C_{13}(T - t)) \leq C_{13}$ for all $\zeta_1 \leq t \leq T$.

Pick $k \geq 1$. Define recursively the following sequence: for $0 \leq t \leq T$,

$$\begin{aligned} \varphi_1(t) &= C_{13}(T - t), \\ \varphi_{n+1}(t) &= \int_t^T \bar{c}(\varphi_n(s)) ds, \quad n = 1, 2, \dots, \infty, \end{aligned}$$

We observe that for all $t \in [\zeta_1, T]$ and $n \geq 2$, we have

$$\mathbb{E} |Y_{n+k}(t) - Y_n(t)|_K^2 \leq \varphi_{n-1}(t) \leq \dots \leq \varphi_1(t). \quad (2.43)$$

This is proved by induction using Lemma 2.8 and Lemma 2.7 as follows. For if $n = 2$, the inequality in (2.43) follows directly from Lemma 2.8. Assume that (2.43) is true for some fixed n with $n \geq 2$. Then

$$\begin{aligned} \mathbb{E} |Y_{n+1+k} - Y_{n+1}(t)|_K^2 &\leq \int_t^T \bar{c}(\mathbb{E} |Y_{n+k}(s) - Y_n(s)|_K^2) ds \\ &\leq \int_t^T \bar{c}(\varphi_{n-1}(s)) ds = \varphi_n(t) \\ &\leq \int_t^T \bar{c}(\varphi_{n-2}(s)) ds = \varphi_{n-1}(t) \end{aligned}$$

and so (2.43) is true for all $n \geq 2$.

On the other hand, by the monotonicity of $\varphi_n(t)$ in n and in t , the sequence $\{\varphi_n(t) : t \in [\zeta_1, T], n \geq 1\}$ attains a limit $\{\varphi(t) : t \in [\zeta_1, T]\}$. Moreover, from the definition of $\varphi_n(t)$ and the dominated convergence theorem, we obtain

$$\varphi(t) = \lim_{n \rightarrow \infty} \int_t^T \bar{c}(\varphi_n(s)) ds = \int_t^T \bar{c}(\varphi(s)) ds, \quad t \in [\zeta_1, T].$$

Thus φ is continuous on $[\zeta_1, T]$.

Therefore by applying Bihari's inequality, we conclude that $\varphi(t) = 0$ for all $t \in [\zeta_1, T]$. This then implies that for any $m \geq 1$ we have

$$\lim_{n \rightarrow \infty} \sup_{t \in [\zeta_1, T]} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \leq \lim_{n \rightarrow \infty} \sup_{t \in [\zeta_1, T]} \varphi_n(t) = \varphi(\zeta_1) = 0, \quad (2.44)$$

which proves our claim (2.42) in the case where $t \in [\zeta_1, T]$.

This together with (2.30) shows that $\{Y_n\}_{n \geq 1}$ is a Cauchy sequence in $L^2_{\mathcal{F}}([\zeta_1, T]; K)$. Call its limit Y . Moreover, from (2.31) and (2.44), it follows that

$$\begin{aligned} & \mathbb{E} \int_{\zeta_1}^T |Z_{n+m}(s) - Z_n(s)|_{L_2(H; K)}^2 ds \\ & \leq C'_{10} \int_{\zeta_1}^T c(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

$\forall n \geq 1$. Thus $\{Z_n\}_{n \geq 1}$ is a Cauchy sequence in $L^2_{\mathcal{F}}(\zeta_1, T; L_2(H; K))$. Call its limit Z . This together with (2.12), (C2) and the convergence of Y_n to Y gives, moreover, that

$$\begin{aligned} & \mathbb{E} \left| \int_t^T e^{A(s-t)} Z_n(s) dW(s) - \int_t^T e^{A(s-t)} Z(s) dW(s) \right|_K^2 = \\ & \mathbb{E} |Y_n(t) - Y(t) + \int_t^T e^{A(s-t)} (f(s, Y_n(s), Z_n(s)) - f(s, Y(s), Z(s))) ds|_K^2 \\ & \leq 2 \mathbb{E} |Y_n(t) - Y(t)|_K^2 + 2 M^2 T \mathbb{E} \int_t^T c(|Y_n(s) - Y(s)|_K^2) ds \\ & + 2 M^2 T k \mathbb{E} \int_t^T |Z_n(s) - Z(s)|_{L_2(H; K)}^2 ds \rightarrow 0. \end{aligned}$$

The convergence of the second term follows from Jensen's inequality and the continuity of the function c . Now, in particular, we can pass the limit as

$n \rightarrow \infty$ in (2.12) and conclude that the pair (Y, Z) solves the original BSEE (2.1) on the interval $[\zeta_1, T]$.

Denote by

$$\zeta_2 := \inf\{\zeta \in [0, T] : \sup_{\zeta \leq t \leq T} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \rightarrow 0, \text{ as } n, m \rightarrow \infty\}.$$

We claim that

$$\sup_{\zeta_2 \leq t \leq T} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \rightarrow 0, \text{ as } n \rightarrow \infty, \forall m \geq 1. \quad (2.45)$$

We now prove this claim. Note first that it is clear from (2.44) that $0 \leq \zeta_2 \leq \zeta_1 < T$. Let $\varepsilon > 0$. Choose $\lambda \in (0, T - \zeta_2)$ such that $C_{13} \lambda < \frac{\varepsilon}{2}$. Since $\bar{c}(0) = 0$, $\exists \theta \in (0, \varepsilon)$ such that $T \bar{c}(\theta) < \frac{\varepsilon}{2}$. Let $N \geq 1$ be sufficiently large so that $\mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 < \theta$ if $\zeta_2 + \lambda \leq t \leq T$, $\forall n \geq N$ and $\forall m \geq 1$. Then by using Lemma 2.7 and Lemma 2.6, we observe that if $n \geq N + 1$, $m \geq 1$ and $t \in [\zeta_2, \zeta_2 + \lambda]$, then

$$\begin{aligned} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 &\leq \int_{\zeta_2}^{\zeta_2 + \lambda} \bar{c}(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\ &\quad + \int_{\zeta_2 + \lambda}^T \bar{c}(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\ &\leq \lambda C_{13} + (T - \zeta_2 - \lambda) \bar{c}(\theta) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon; \end{aligned}$$

recall here that $C_{13} := \bar{c}(4 C_1)$. Thus we obtain

$$\sup_{\zeta_2 \leq t \leq T} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 < \varepsilon, \quad \forall n \geq N + 1 \text{ and } m \geq 1,$$

proving the claim (2.45).

The proof of the claim (2.42) finishes and so the theorem, as explained earlier, when proving that $\zeta_2 = 0$, which is the case as will be shown now.

Suppose otherwise that $\zeta_2 > 0$. By using claim (2.45) we can choose a sequence of decreasing numbers $\{a_n\}_{n \geq 1}$ such that $a_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$\sup_{\zeta_2 \leq t \leq T} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 < a_n \quad \forall n \geq 1. \quad (2.46)$$

If $0 \leq t \leq \zeta_2$ and $n \geq 1$, then by Lemma 2.7 and (2.46), we derive that

$$\begin{aligned}
\mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 &\leq \int_t^T \bar{c}(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\
&\leq \left(\int_t^{\zeta_2} + \int_{\zeta_2}^T \right) \bar{c}(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds \\
&\leq \int_t^{\zeta_2} \bar{c}(\mathbb{E} |Y_{n+m-1}(s) - Y_{n-1}(s)|_K^2) ds + T \bar{c}(a_{n-1}) \\
&\leq (\zeta_2 - t) C_{13} + T \bar{c}(a_{n-1}). \tag{2.47}
\end{aligned}$$

Pick $\delta \in (0, \zeta_2)$ and $j \geq 1$ such that

$$C_{13} \delta + T \bar{c}(a_j) \leq 4 C_1. \tag{2.48}$$

Define a sequence of functions $\{\psi_k(t), \zeta_2 - \delta \leq t \leq \zeta_2, k \geq 1\}$ by

$$\begin{aligned}
\psi_1(t) &= C_{13} \delta + T \bar{c}(a_j) \leq 4 C_1, \\
\psi_{k+1}(t) &= T \bar{c}(a_{j+1}) + \int_t^{\zeta_2} \bar{c}(\psi_k(s)) ds, \quad k \geq 1.
\end{aligned}$$

Let us now admit for a moment that for fixed $l \geq 1$, if $\zeta_2 - \delta \leq t \leq \zeta_2$ and $k \geq 1$, we have

$$\mathbb{E} |Y_{l+j+k}(t) - Y_{j+k}(t)|_K^2 \leq \psi_k(t) \leq \cdots \leq \psi_1(t). \tag{2.49}$$

Then, for each $t \in [\zeta_2 - \delta, \zeta_2]$, $\psi_k(t)$ attains a limit as $k \rightarrow \infty$, $\psi(t)$, say, defined on $[\zeta_2 - \delta, \zeta_2]$. But

$$\begin{aligned}
\psi(t) &= \lim_{k \rightarrow \infty} \psi_{k+1}(t) = \lim_{k \rightarrow \infty} \left[T \bar{c}(a_{j+k}) + \int_t^{\zeta_2} \bar{c}(\psi_k(s)) ds \right] \\
&= \int_t^{\zeta_2} \bar{c}(\psi(s)) ds,
\end{aligned}$$

for all $t \in [\zeta_2 - \delta, \zeta_2]$. Therefore $\psi(t) = 0$ on $\zeta_2 - \delta \leq t \leq \zeta_2$, by using Bihari's inequality. It follows then from (2.49) that

$$\sup_{\zeta_2 - \delta \leq t \leq \zeta_2} \mathbb{E} |Y_{l+j+k}(t) - Y_{j+k}(t)|_K^2 \leq \psi_k(\zeta_2 - \delta) \rightarrow 0, \tag{2.50}$$

as $k \rightarrow \infty$.

Since $l \geq 1$ is arbitrary then (2.50) implies that

$$\sup_{\zeta_2 - \delta \leq t \leq \zeta_2} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \rightarrow 0,$$

as $n \rightarrow \infty$, $\forall m \geq 1$. This together with (2.45) gives

$$\sup_{\zeta_2 - \delta \leq t \leq T} \mathbb{E} |Y_{n+m}(t) - Y_n(t)|_K^2 \rightarrow 0,$$

contradicting the definition of ζ_2 . Thus $\zeta_2 = 0$, as required.

Finally, it remains only to show (2.49). Let $t \in [\zeta_2 - \delta, \zeta_2]$. If $k = 1$, then

$$\begin{aligned} \mathbb{E} |Y_{l+j+1}(t) - Y_{j+1}(t)|_K^2 &\leq (\zeta_2 - t) C_{13} + T \bar{c}(a_j) \\ &= \psi_1(t). \end{aligned}$$

Moreover, for $k = 2$, as done for (2.47), we get

$$\begin{aligned} &\mathbb{E} |Y_{l+j+2}(t) - Y_{j+2}(t)|_K^2 \\ &\leq T \bar{c}(a_{j+1}) + \int_t^{\zeta_2} \bar{c}(\mathbb{E} |Y_{l+j+1}(s) - Y_{j+1}(s)|_K^2) ds \\ &\leq T \bar{c}(a_{j+1}) + \int_t^{\zeta_2} \bar{c}(\psi_1(s)) ds = \psi_2(t) \\ &\leq T \bar{c}(a_j) + C_{13} (\zeta_2 - t) = \psi_1(t), \end{aligned}$$

by using Lemma 2.7, (2.48) and the definition of $C_{13} = \bar{c}(4 C_1)$. The last inequality comes from $a_{j+1} \leq a_j$ and \bar{c} being nondecreasing. Thus (2.49) holds for $k = 1, 2$.

Assume that (2.49) is true for some k with $k \geq 2$. Then, as for the case $k = 2$, by using Lemma 2.7 and (2.46), we get that

$$\begin{aligned} &\mathbb{E} |Y_{l+j+k+1}(t) - Y_{j+k+1}(t)|_K^2 \\ &\leq T \bar{c}(a_{j+k}) + \int_t^{\zeta_2} \bar{c}(\mathbb{E} |Y_{l+j+k}(s) - Y_{j+k}(s)|_K^2) ds \\ &= T \bar{c}(a_{j+k}) + \int_t^{\zeta_2} \bar{c}(\psi_k(s)) ds = \psi_{k+1}(t) \\ &\leq T \bar{c}(a_{j+k-1}) + \int_t^{\zeta_2} \bar{c}(\psi_{k-1}(s)) ds = \psi_k(t). \end{aligned}$$

Hence (2.49) holds for $k + 1$ as well. By induction (2.49) therefore holds for every $k \geq 1$.

Uniqueness : Suppose that (Y, Z) and (Y', Z') are two solutions to (2.1). Then

$$\begin{aligned} Y(t) - Y'(t) &= \int_t^T e^{A(s-t)} [f(s, Y(s), Z(s)) - f(s, Y'(s), Z'(s))] ds \\ &\quad - \int_t^T e^{A(s-t)} [Z(s) - Z'(s)] dW(s). \end{aligned} \quad (2.51)$$

By letting $\tilde{Y}(t) = Y(t) - Y'(t)$, $\tilde{Z}(t) = Z(t) - Z'(t)$ and $\tilde{f}(t) = f(t, Y(t), Z(t)) - f(t, Y'(t), Z'(t))$, equation (2.51) becomes one of the sort of equation (2.3) in Lemma 2.2 since hypotheses (C1) and (C2) and Jensen's inequality give

$$\mathbb{E} \int_0^T |\tilde{f}(s)|_K^2 ds \leq c(\mathbb{E} \int_0^T |\tilde{Y}(s)|_K^2 ds) + \mathbb{E} \int_0^T k |\tilde{Z}(s)|_{L_2(H;K)}^2 ds < \infty,$$

as $c(x) \leq a x + b$, for some $a, b > 0$. Thus, as in Lemma 2.2, we have

$$\begin{aligned} \mathbb{E} |\tilde{Y}(t)|_K^2 &\leq 2M^2 (T-t) \mathbb{E} \int_t^T |\tilde{f}(s)|_K^2 ds \\ &\leq 2M^2 (T-t) \mathbb{E} \int_t^T c(|\tilde{Y}(s)|_K^2) ds \\ &\quad + 2M^2 (T-t) k \mathbb{E} \int_t^T |\tilde{Z}(s)|_{L_2(H;K)}^2 ds \end{aligned} \quad (2.52)$$

and

$$\begin{aligned} \mathbb{E} \int_t^T |\tilde{Z}(s)|_{L_2(H;K)}^2 ds &\leq 8M^2 (T-t) \mathbb{E} \int_t^T |\tilde{f}(s)|_K^2 ds \\ &\leq 8M^2 (T-t) \mathbb{E} \int_t^T c(|\tilde{Y}(s)|_K^2) ds \\ &\quad + 8M^2 (T-t) k \mathbb{E} \int_t^T |\tilde{Z}(s)|_{L_2(H;K)}^2 ds. \end{aligned} \quad (2.53)$$

As we did in the proof of Lemma 2.6, one can easily use the trick of partitioning the interval $[0, T]$ for (2.52) and for (2.53), using a fixed small

scale η such that $0 \leq \eta < T$ with $16 M^2 \eta k < 1$. We then get eventually that if, for example, $t \geq 0$ and $T - (l + 1) \eta \leq t \leq T - l \eta$ for $0 \leq l \leq q$, then $\exists C(l) > 0$, a constant that possibly depends on l , such that

$$\mathbb{E} |\tilde{Y}(t)|_K^2 \leq C(l) \int_t^T c(\mathbb{E} |\tilde{Y}(s)|_K^2) ds \quad (2.54)$$

and

$$\mathbb{E} \int_0^T |\tilde{Z}(s)|_{L_2(H;K)}^2 ds \leq C(l) \int_0^T c(\mathbb{E} |\tilde{Y}(s)|_K^2) ds. \quad (2.55)$$

Recall here q is the smallest integer such that $q \geq \frac{T}{\eta}$.

Therefore by summing over $0 \leq l \leq q$ in (2.54) and applying Bihari's inequality afterwards, we get that $\tilde{Y}(t) = 0$ a.s. $\forall t \in [0, T]$.

Similarly, take the sum over l in (2.55) and use this resulting uniqueness of Y to conclude that $\mathbb{E} \int_0^T |\tilde{Z}(s)|_{L_2(H;K)}^2 ds = 0$. ■

2.2 Regularities of Solutions of BSEEs

In this section we discuss the continuity in t of the solution Y of the BSEE (2.1). Then we provide a priori estimates for $\mathbb{E} \sup_{t \in [0, T]} |Y(t)|_K^2$ and for $\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T e^{A(s-t)} Z(s) dW(s) \right|_K^2$, and generalise this to higher order moments.

The results of this section are also valid for the case when replacing A by a time dependent operator, which is the case discussed in Section 4 below. For the proofs, one uses the same reasoning we will be using below for the semigroup case. See, for example, the change of proof made to Lemma 2.23 below, which generalises Lemma 2.2. See also Proposition 2.22 below which will be taken as a generalisation of the next result.

Proposition 2.10 *Let $f \in L_{\mathcal{F}}^2(0, T; K)$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$. Then the solution Y of the following BSEE*

$$\begin{cases} -dY(t) = A Y(t) dt + f(t) dt - Z(t) dW(t), \\ Y(t) = \xi. \end{cases} \quad (2.56)$$

has a version which is continuous almost surely as a process in K .

Proof. Note that the solution of (2.56) is given by

$$Y(t) = e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} f(s) ds - \int_t^T e^{A(s-t)} Z(s) dW(s). \quad (2.57)$$

For convenience, we shall show that each term of (2.57) is continuous¹ in t .

Since $\{\mathcal{F}_t : 0 \leq t \leq T\}$ is the Wiener filtration, then according to Theorem 1.2, $\mathbb{E}[\xi | \mathcal{F}_t]$ is continuous in t , for each $t \in [0, T]$. Therefore $\mathbb{E}[e^{A(T-t)} \xi | \mathcal{F}_t] = e^{A(T-t)} \mathbb{E}[\xi | \mathcal{F}_t]$ is also continuous in t .

On the other hand, for the same reason, $\mathbb{E}[e^{A(s-t)} f(s) | \mathcal{F}_t]$ is continuous in t , for each $t \leq s \leq T$. Moreover,

$$\mathbb{E}\left[\int_t^T e^{A(s-t)} f(s) ds \mid \mathcal{F}_t\right] = \int_t^T e^{A(s-t)} \mathbb{E}[f(s) | \mathcal{F}_t] ds.$$

Thus $\mathbb{E}\left[\int_t^T e^{A(s-t)} f(s) ds \mid \mathcal{F}_t\right]$ is continuous in t .

Since

$$Y(t) = \mathbb{E}[e^{A(T-t)} \xi | \mathcal{F}_t] + \mathbb{E}\left[\int_t^T e^{A(s-t)} f(s) ds \mid \mathcal{F}_t\right],$$

then Y is continuous a.s.

As a result from this and from (2.57) we obtain the continuity of $\int_t^T e^{A(s-t)} Z(s) dW(s)$. ■

We then conclude immediately the following continuity result to the solution of the BSEE (2.1).

Corollary 2.11 *The solution $Y(t)$, $0 \leq t \leq T$, of the BSEE (2.1) given by the form (2.2) and the integral $\int_t^T e^{A(s-t)} Z(s) dW(s)$ are both continuous almost surely in t .*

¹i.e. has a continuous version

Proposition 2.12 *Under the same conditions as in Theorem 2.1, the following estimates hold*

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y(t)|_K^2 < \infty, \quad (2.58)$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T e^{A(s-t)} Z(s) dW(s) \right|_K^2 < \infty, \quad (2.59)$$

for the solution (Y, Z) of (2.1).

Proof. Since Y is given in the following form

$$\begin{aligned} Y(t) = & e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds \\ & - \int_t^T e^{A(s-t)} Z(s) dW(s), \end{aligned}$$

then we derive that

$$\begin{aligned} |Y(t)|_K &= \left| \mathbb{E} \left[e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds \mid \mathcal{F}_t \right] \right|_K \\ &\leq \mathbb{E} \left[M |\xi|_K + M \int_t^T |f(s, Y(s), Z(s))|_K ds \mid \mathcal{F}_t \right] \\ &\leq M \mathbb{E} \left[|\xi|_K + \int_0^T |f(s, Y(s), Z(s))|_K ds \mid \mathcal{F}_t \right] \text{ a.s.} \end{aligned}$$

Since the right hand side of this last inequality is a continuous martingale, then, by using Doob's inequality for martingales, we get that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |Y(t)|_K^2 &\leq \\ &2 M^2 \mathbb{E} \sup_{0 \leq t \leq T} \left(\mathbb{E} \left[|\xi|_K^2 + T \int_0^T |f(s, Y(s), Z(s))|_K^2 ds \mid \mathcal{F}_t \right] \right) \\ &\leq 8 M^2 \left(\mathbb{E} |\xi|_K^2 + T \mathbb{E} \int_0^T |f(s, Y(s), Z(s))|_K^2 ds \right). \end{aligned}$$

But (C1), (C2) and Jensen's inequality imply that

$$\begin{aligned} \mathbb{E} \int_0^T |f(s, Y(s), Z(s))|_K^2 ds &\leq 2 c \left(\mathbb{E} \int_0^T |Y(s)|_K^2 ds \right) \\ &+ 2 k \mathbb{E} \int_0^T |Z(s)|_{L_2(H;K)}^2 ds + 2 \mathbb{E} \int_0^T |f(s, 0, 0)|_K^2 ds < \infty. \end{aligned}$$

Hence (2.58) follows.

Finally, since

$$\int_t^T e^{A(s-t)} Z(s) dW(s) = e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds - Y(t),$$

then

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T e^{A(s-t)} Z(s) dW(s) \right|_K^2 \leq \\ & 3 M^2 \mathbb{E} |\xi|_K^2 + 3 T M^2 \mathbb{E} \int_0^T |f(s)|_K^2 ds + \mathbb{E} \sup_{0 \leq t \leq T} |Y(t)|_K^2 < \infty, \end{aligned}$$

which proves (2.59). ■

Remark 2.13 *In Proposition 2.12 we estimated only the second order moments of the solution (Y, Z) . However, as seen from the proof, one can easily obtain the same result for higher order moments. In particular, for $2 < p < \infty$ if $\mathbb{E} |\xi|_K^p + \mathbb{E} \int_0^T |Y(s)|_K^p ds + \mathbb{E} \int_0^T |Z(s)|_{L_2(H;K)}^p ds < \infty$, then*

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y(t)|_K^p < \infty,$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T e^{A(s-t)} Z(s) dW(s) \right|_K^p < \infty.$$

2.3 Weak Solutions, Weak Mild Solutions and Strong Solutions of BSEEs

In Theorem 2.1 we saw that a mild solution to (2.1) exists and is unique. In this section we will discuss the notions of weak, weak mild and strong solutions of the BSEE (2.1). We shall show that the first two types of these solutions are equivalent and both of them are equivalent to the mild solution.

The existence and uniqueness of these types of solutions then follow from Theorem 2.1 above, under the same conditions. We close the section by showing that a strong solution of (2.1) also exists and is unique under extra condition; see Theorem(s) (2.18, 2.19) below. Our business here is similar to that in [27] and [21] in the forward case.

Let us now recall some basic properties of C_0 -semigroups; see [24] for more details.

First let A be a densely defined closed operator on the space K ; denote its domain by $\mathcal{D}(A)$. Then $\mathcal{D}(A)$ is a Hilbert space with respect to the graph norm $|y|_{\mathcal{D}(A)}^2 := |y|_K^2 + |A y|_K^2$, continuously and densely embedded in K . Moreover, if $y \in \mathcal{D}(A)$ then $e^{At} y \in \mathcal{D}(A)$ and $\frac{\partial}{\partial t} e^{At} y = e^{At} A y = A e^{At} y$.

Let $\{e^{A^*t} : 0 \leq t \leq T\}$ denote the adjoint semigroup of $\{e^{At} : 0 \leq t \leq T\}$, which is also a C_0 -semigroup with an infinitesimal generator A^* , the adjoint of A . In particular, $\mathcal{D}(A^*)$ is dense in K .

Let us now introduce the following definitions.

Definition 2.14 A $K \times L_2(H; K)$ -valued process (Y, Z) is called a strong solution to BSEE (2.1) if

- (i) $Y \in L_{\mathcal{F}}^2(0, T; K)$ and $Z \in L_{\mathcal{F}}^2(0, T; L_2(H; K))$,
- (ii) $Y(t) \in \mathcal{D}(A)$ a.s. for a.e. $0 \leq t \leq T$ and $AY(\cdot) \in L^1([0, T]; K)$ a.s.
- (iii) For every $t \in [0, T]$ we have a.s.

$$Y(t) = \xi + \int_t^T (AY(s) + f(s, Y(s), Z(s))) ds - \int_t^T Z(s) dW(s). \quad (2.60)$$

Definition 2.15 A $K \times L_2(H; K)$ -valued process (Y, Z) is called a weak solution to BSEE (2.1) if

- (i) $Y \in L_{\mathcal{F}}^2(0, T; K)$ and $Z \in L_{\mathcal{F}}^2(0, T; L_2(H; K))$,
- (ii) For every $t \in [0, T]$ and $\forall \rho \in \mathcal{D}(A^*)$ we have a.s.

$$\begin{aligned}
\langle Y(t), \rho \rangle_K &= \langle \xi, \rho \rangle_K + \int_t^T \langle Y(s), A^* \rho \rangle_K ds \\
&+ \int_t^T \langle f(s, Y(s), Z(s)), \rho \rangle_K ds - \int_t^T \langle Z(s) dW(s), \rho \rangle_K. \quad (2.61)
\end{aligned}$$

Definition 2.16 A $K \times L_2(H; K)$ -valued process (Y, Z) is called a weak mild solution to BSEE (2.1) if

- (i) $Y \in L^2_{\mathcal{F}}(0, T; K)$ and $Z \in L^2_{\mathcal{F}}(0, T; L_2(H; K))$,
- (ii) For every $t \in [0, T]$ and $\forall \rho \in K$ we have a.s.

$$\begin{aligned}
\langle Y(t), \rho \rangle_K &= \langle \xi, e^{A^*(T-t)} \rho \rangle_K \\
&+ \int_t^T \langle e^{A^*(s-t)} \rho, f(s, Y(s), Z(s)) \rangle_K ds \\
&- \int_t^T \langle e^{A^*(s-t)} \rho, Z(s) dW(s) \rangle_K. \quad (2.62)
\end{aligned}$$

The following theorem tells us the relationships between such solutions.

Theorem 2.17 Consider the following cases:

- (1) (Y, Z) is a strong solution to BSEE (2.1).
- (2) (Y, Z) is a weak solution to BSEE (2.1).
- (3) (Y, Z) is a mild solution to BSEE (2.1).
- (4) (Y, Z) is a weak mild solution to BSEE (2.1).

Then (1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4).

Proof. (1) \Rightarrow (2) : Let $\rho \in \mathcal{D}(A^*)$. Then from (2.60) we have

$$\begin{aligned}
\langle Y(t), \rho \rangle_K &= \langle \xi, \rho \rangle_K + \int_t^T \langle Y(s), A^* \rho \rangle_K ds \\
&+ \int_t^T \langle f(s, Y(s), Z(s)), \rho \rangle_K ds \\
&- \int_t^T \langle Z(s) dW(s), \rho \rangle_K,
\end{aligned}$$

i.e. (Y, Z) is a weak solution to (2.1).

(3) \Rightarrow (4) : Let $\rho \in K$. Then from (2.2) we have

$$\begin{aligned}
\langle Y(t), \rho \rangle_K &= \langle e^{A(T-t)} \xi, \rho \rangle_K \\
&+ \int_t^T \langle e^{A(s-t)} f(s, Y(s), Z(s)), \rho \rangle_K ds \\
&- \int_t^T \langle e^{A(s-t)} Z(s) dW(s), \rho \rangle_K \\
&= \langle \xi, e^{A^*(T-t)} \rho \rangle_K \\
&+ \int_t^T \langle e^{A^*(s-t)} \rho, f(s, Y(s), Z(s)) \rangle_K ds \\
&- \int_t^T \langle e^{A^*(s-t)} \rho, Z(s) dW(s) \rangle_K,
\end{aligned}$$

i.e. (Y, Z) is a weak mild solution to (2.1).

(4) \Rightarrow (3) : By using (2.62) the following identity holds *a.s.*

$$\begin{aligned}
&\langle Y(t) - e^{A(T-t)} \xi - \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds + \\
&\int_t^T e^{A(s-t)} Z(s) dW(s), \rho \rangle_K = 0,
\end{aligned}$$

for all $\rho \in K$ and for all $t \in [0, T]$. This implies that (Y, Z) satisfies (2.2) and so is a mild solution.

It remains to show that (2) \Leftrightarrow (4).

(2) \Rightarrow (4). It is sufficient to prove that the following equation holds for each $t \in [0, T]$ and $\forall \rho \in \mathcal{D}(A^{*2})$ since $\mathcal{D}(A^{*2})$ is dense in K (cf. [45]) :

$$\begin{aligned}
\langle Y(t), \rho \rangle_K &= \langle \xi, e^{A^*(T-t)} \rho \rangle_K \\
&+ \int_t^T \langle e^{A^*(s-t)} \rho, f(s, Y(s), Z(s)) \rangle_K ds \\
&- \int_t^T \langle e^{A^*(s-t)} \rho, Z(s) dW(s) \rangle_K.
\end{aligned} \tag{2.63}$$

Notice that $\forall \tilde{\rho} \in \mathcal{D}(A^*)$ we have

$$\begin{aligned} \langle Y(r), \tilde{\rho} \rangle_K &= \langle \xi, \tilde{\rho} \rangle_K + \int_r^T \langle Y(s), A^* \tilde{\rho} \rangle_K ds \\ &+ \int_r^T \langle f(s, Y(s), Z(s)), \tilde{\rho} \rangle_K ds - \int_r^T \langle Z(s) dW(s), \tilde{\rho} \rangle_K. \end{aligned} \quad (2.64)$$

For fixed $r \in [0, T]$, take $\tilde{\rho} := e^{A^*(r-t)} A^* \rho$. Then $\tilde{\rho} \in \mathcal{D}(A^*)$. Apply (2.64) to this $\tilde{\rho}$ and integrate with respect to r from t to T to get the following

$$\begin{aligned} \int_t^T \langle Y(r), e^{A^*(r-t)} A^* \rho \rangle_K dr &= \\ \int_t^T \langle \xi, e^{A^*(r-t)} A^* \rho \rangle_K dr &+ \\ + \int_t^T \int_r^T \langle Y(s), A^* e^{A^*(r-t)} A^* \rho \rangle_K ds dr &+ \\ + \int_t^T \int_r^T \langle A^* e^{A^*(r-t)} A^* \rho, f(s, Y(s), Z(s)) \rangle_K ds dr &- \\ - \int_t^T \int_r^T \langle e^{A^*(r-t)} A^* \rho, Z(s) dW(s) \rangle_K dr & \\ =: I_1 + I_2 + I_3 + I_4. & \end{aligned} \quad (2.65)$$

But by simple calculation

$$I_1 = \langle \xi, e^{A^*(T-t)} \rho - \rho \rangle_K. \quad (2.66)$$

By using Fubini's theorem we get

$$\begin{aligned} I_2 &= \int_t^T \int_r^T \langle Y(s), A^* e^{A^*(r-t)} A^* \rho \rangle_K ds dr \\ &= \int_t^T \int_t^s \langle Y(s), A^* e^{A^*(r-t)} A^* \rho \rangle_K dr ds \\ &= \int_t^T \langle Y(s), e^{A^*(T-t)} A^* \rho \rangle_K ds \\ &\quad - \int_t^T \langle Y(s), A^* \rho \rangle_K ds. \end{aligned} \quad (2.67)$$

Similarly by using Fubini's theorem, we get

$$\begin{aligned}
I_3 &= \int_t^T \int_t^s \langle A^* e^{A^*(r-t)} A^* \rho, f(s, Y(s), Z(s)) \rangle_K ds dr \\
&= \int_t^T \langle e^{A^*(s-t)} \rho, f(s, Y(s), Z(s)) \rangle_K ds \\
&\quad - \int_t^T \langle \rho, f(s, Y(s), Z(s)) \rangle_K ds.
\end{aligned} \tag{2.68}$$

On the other hand,

$$\begin{aligned}
I_4 &= - \langle \rho, \int_t^T A e^{A(r-t)} M(r) dr \rangle_K \\
&= \langle \rho, M(t) - \int_t^T e^{A(r-t)} Z(r) dW(r) \rangle_K,
\end{aligned}$$

by using integration by parts, where $M(r) := \int_r^T Z(s) dW(s)$. In particular,

$$I_4 = \int_t^T \langle \rho, Z(s) dW(s) \rangle_K - \int_t^T \langle e^{A^*(s-t)} \rho, Z(s) dW(s) \rangle_K. \tag{2.69}$$

By substituting (2.66) – (2.69) into (2.65), using (2.64) (for $\tilde{\rho} = \rho$) and re-arranging terms, we obtain (2.63).

(4) \Rightarrow (2) : Let $\rho \in \mathcal{D}(A^*)$. Apply (2.63) for $A^* \rho$ and integrate from t to T with respect to s to get that

$$\begin{aligned}
&\int_t^T \langle Y(s), A^* \rho \rangle_K ds = \int_t^T \langle \xi, e^{A^*(T-s)} A^* \rho \rangle_K ds \\
&+ \int_t^T \int_s^T \langle e^{A^*(r-s)} A^* \rho, f(r, Y(r), Z(r)) \rangle_K dr ds \\
&- \int_t^T \int_s^T \langle e^{A^*(r-s)} A^* \rho, Z(r) dW(r) \rangle_K ds \\
&=: I_5 + I_6 + I_7.
\end{aligned} \tag{2.70}$$

As we did in the preceding case, we find that

$$I_5 = \langle \xi, e^{A^*(T-t)} A^* \rho \rangle_K - \langle \xi, \rho \rangle_K. \tag{2.71}$$

By using Fubini's theorem we derive that

$$\begin{aligned}
I_6 &= \int_t^T \langle \int_t^r e^{A^*(r-s)} A^* \rho \, ds, f(r, Y(r), Z(r)) \rangle_K \, dr \\
&= \int_t^T \langle e^{A(r-t)} \rho - \rho, f(r, Y(r), Z(r)) \rangle_K \, dr \\
&= \int_t^T \langle e^{A(r-s)} f(r, Y(r), Z(r)), \rho \rangle_K \, dr \\
&\quad - \int_t^T \langle \rho, f(r, Y(r), Z(r)) \rangle_K \, dr.
\end{aligned} \tag{2.72}$$

Similarly by using Fubini's theorem, we get

$$\begin{aligned}
I_7 &= - \int_t^T \langle \int_t^r e^{A^*(r-s)} A^* \rho \, ds, Z(r) \, dW(r) \rangle_K \\
&= - \int_t^T \langle e^{A^*(r-t)} \rho, Z(r) \, dW(r) \rangle_K \\
&\quad - \int_t^T \langle \rho, Z(r) \, dW(r) \rangle_K.
\end{aligned} \tag{2.73}$$

By substituting (2.71) – (2.73) into (2.70) and using (2.63) we obtain

$$\begin{aligned}
\langle Y(t), \rho \rangle_K &= \langle \xi, \rho \rangle_K + \int_t^T \langle Y(s), A^* \rho \rangle_K \, ds \\
&\quad + \int_t^T \langle \rho, f(s, Y(s), Z(s)) \rangle_K \, ds - \int_t^T \langle \rho, Z(s) \, dW(s) \rangle_K,
\end{aligned}$$

proving that (Y, Z) is a weak solution to (2.1). ■

Theorem 2.1 and Theorem 2.17 show the existence and the uniqueness of mild, weak and weak mild solutions to the BSEE (2.1) under the conditions (C1) and (C2). Moreover, by using the same method as in Proposition 2.10, one can show the continuity in t of such solutions; alternatively this can also be seen directly from the definitions.

On the other hand, for a strong solution to exist we will need extra conditions as the following two theorems show.

Theorem 2.18 *If a weak solution (Y, Z) of (2.1) satisfies $Y(t) \in \mathcal{D}(A)$ a.s. a.e. $t \in [0, T]$ and $A Y(\cdot) \in L^1([0, T]; K)$ a.s., then (Y, Z) is a strong solution to (2.1).*

Proof. Note that under our conditions

$$\int_t^T \langle Y(s), A^* \rho \rangle_K ds = \langle \int_t^T A Y(s) ds, \rho \rangle_K$$

holds for every $\rho \in \mathcal{D}(A^*)$ and since $\mathcal{D}(A^*)$ is dense in K , then (2.61) implies that for every $t \in [0, T]$ we have a.s.

$$Y(t) = \int_t^T A Y(s) ds + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s),$$

i.e. (Y, Z) is a strong solution to (2.1). ■

In the following theorem we prove the existence and uniqueness of strong solutions to (2.1).

Theorem 2.19 *Assume that (2.1) has a unique mild solution (Y, Z) (e.g. under (C1) and (C2)). Assume, moreover, that*

(i) $\xi \in \mathcal{D}(A)$ and $\mathbb{E} |\xi|_K^2 < \infty$.

(ii) $e^{A(s-t)} f(s, Y(s), Z(s)) \in \mathcal{D}(A)$, $e^{A(s-t)} Z(s) h \in \mathcal{D}(A)$ a.s., $\forall s \in (t, T]$ and $\forall h \in H$.

(iii) $\mathbb{E} [\int_0^T \int_t^T |A e^{A(s-t)} f(s, Y(s), Z(s))|_K ds dt] < \infty$

(iv) $\mathbb{E} [\int_0^T \int_t^T |A e^{A(s-t)} Z(s)|_{L_2(H;K)}^2 ds dt] < \infty$.

Then (2.1) has a unique strong solution which is a continuous version of the mild solution (Y, Z) .

Proof. Recall that for every $t \in [0, T]$, $Y(t)$ is given by

$$Y(t) = e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds - \int_t^T e^{A(s-t)} Z(s) dW(s). \quad (2.74)$$

From (i) we see that $e^{A(T-t)} \xi \in \mathcal{D}(A)$ and from (ii) we get that $\int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds \in \mathcal{D}(A)$ a.s. for a.e. $t \in [0, T]$, and $\int_t^T e^{A(s-t)} Z(s) dW(s) \in \mathcal{D}(A)$ a.s. for a.e. $t \in [0, T]$. In particular, $Y(t) \in \mathcal{D}(A)$ for a.e. $t \in [0, T]$.

Moreover, (iii) and (iv) give that

$$A \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds = \int_t^T A e^{A(s-t)} f(s, Y(s), Z(s)) ds ,$$

$0 \leq t \leq T$, and

$$A \int_t^T e^{A(s-t)} Z(s) dW(s) = \int_t^T A e^{A(s-t)} Z(s) dW(s) , \quad 0 \leq t \leq T.$$

Therefore we conclude that

$$\begin{aligned} A Y(t) &= A e^{A(T-t)} \xi + \int_t^T A e^{A(s-t)} f(s, Y(s), Z(s)) ds \\ &\quad - \int_t^T A e^{A(s-t)} Z(s) dW(s) , \quad 0 \leq t \leq T, \end{aligned}$$

In particular,

$$A Y(t) = \mathbb{E} \left[A e^{A(T-t)} \xi + \int_t^T A e^{A(s-t)} f(s, Y(s), Z(s)) ds \mid \mathcal{F}_t \right] ,$$

$0 \leq t \leq T$, which together with Jensen's inequality, (i) and (ii) implies that $\mathbb{E} \int_0^T |A Y(t)|_K dt < \infty$. Hence $\int_0^T |A Y(t)|_K dt < \infty$ a.s. So now we can apply Theorem 2.18 to see that the right hand side of (2.74) is a strong solution to (2.1).

Uniqueness of strong solutions follows from the fact that they are versions of the unique mild solution, cf. Theorem 2.17. ■

2.4 Evolution Solutions of BSEEs

In this section we let the operator A appearing in the BSEE (2.1) to depend in a measurable way on time, i.e such that $[0, T] \ni t \mapsto A(t) y \in K$ is Borel

measurable, for all $y \in K$. We shall generalise briefly the results in Section 1 and Section 2 to this time dependent case, called below the *evolution case*. A brief discussion about the type of solutions to BSEE (evolution case), see (2.75) below, is given with some of the relationship between such solutions.

Let us now recall some of the definitions which we will need.

Definition 2.20² *A two parameter family of bounded linear operators $\{U(s, t), 0 \leq t \leq s \leq T\}$ on K is called an evolution system if the following holds*

- (i) $U(s, s) = I, 0 \leq s \leq T,$
- (ii) $U(s, r) U(r, t) = U(s, t), 0 \leq t \leq r \leq s,$
- (iii) $(s, t) \mapsto U(s, t)$ is strongly continuous for $0 \leq t \leq s.$

We call the mapping U an evolution operator.

Definition 2.21 *A strong evolution operator is an evolution operator $U(s, t), 0 \leq t \leq s \leq T,$ for which there exists a closed and densely defined linear operator $A(s),$ with domain $\mathcal{D}(A(s)), s \geq 0,$ such that*

$$U(s, t)(\mathcal{D}(A(t))) \subset \mathcal{D}(A(s)), \quad s > t,$$

and

$$\frac{\partial}{\partial s} U(s, t) y = A(s) U(s, t) y, \quad s > t, \quad y \in \mathcal{D}(A(t)).$$

The family $A(s), 0 \leq s \leq T,$ is called the infinitesimal generator of $U(s, t), 0 \leq t \leq s \leq T.$

In particular, each $A(s)$ is a generator of a C_0 -semigroup on K for all $s \in [0, T]$. An example of which is the C_0 -semigroup $\{e^{As}, 0 \leq s \leq T\}$ of infinitesimal generator A , which defines a strong evolution operator by $U(s, t) := e^{A(s-t)}$ with infinitesimal generator $A(s) = A$ for all $s.$

² T here does not have to be finite for this definition to work, however we require T to be finite in our study of BSDEs and BSEEs.

All $\{A(s), s \geq 0\}$ will be assumed to be closed and densely defined linear operators, which generate a strong evolution operator $U(s, t)$, $0 \leq t \leq s \leq T$. For more information on such operators see [54], [45] or [21].

We assume also that $[0, T]^2 \rightarrow L(K)$, $(s, t) \mapsto U(s, t)$ is measurable.

Consider now the following BSEE (evolution case)

$$\begin{cases} -dY(t) = (A(t) Y(t) + f(t, Y(t), Z(t))) dt - Z(t) dW(t), & 0 \leq t \leq T \\ Y(T) = \xi, \end{cases} \quad (2.75)$$

where f and ξ are as in Section 1 and satisfy (C1) and (C2), and W is a cylindrical Wiener process on H .

An *evolution* solution of (2.75) is a pair $(Y, Z) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$, such that the following equality holds *a.s.*

$$\begin{aligned} Y(t) &= U(T, t) \xi + \int_t^T U(s, t) f(s, Y(s), Z(s)) ds \\ &\quad - \int_t^T U(s, t) Z(s) dW(s), \quad 0 \leq t \leq T. \end{aligned} \quad (2.76)$$

Our aim is to show that equation (2.75) has a unique evolution solution. Before doing so let us present the following property of the solution of this equation.

Proposition 2.22 *If (Y, Z) is a solution of the BSEE (2.75), then Y has a continuous version.*

Proof. Note that

$$Y(t) = \mathbb{E} \left[U(T, t) \xi + \int_t^T U(s, t) f(s, Y(s), Z(s)) ds \mid \mathcal{F}_t \right] \text{ a.s. ,}$$

for all $t \in [0, T]$. Since $t \mapsto U(T, t) \xi$ is continuous a.s. on $[0, T]$, then so is $\mathbb{E} [U(T, t) \xi \mid \mathcal{F}_t]$, as a consequence of Theorem 1.2.

Now by arguing as we did in the proof of Proposition 2.10, see also Corollary 2.11, we can show easily that $\mathbb{E} \left[\int_t^T U(s, t) f(s, Y(s), Z(s)) ds \mid \mathcal{F}_t \right]$ is

continuous a.s. in t , for all $t \in [0, T]$. Consequently $Y(t)$ is continuous a.s. in t , for all $t \in [0, T]$. ■

We conclude also from this result and from (2.76) that $\int_t^T U(s, t) Z(s) dW(s)$ is continuous a.s. for all $t \in [0, T]$.

Let us now study a simplified version of (2.75) in the following lemma.

Lemma 2.23 *If $f \in L^2_{\mathcal{F}}(0, T; K)$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$, then there exists a unique pair $(Y, Z) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$ such that*

$$\begin{aligned} Y(t) = & U(T, t) \xi + \int_t^T U(s, t) f(s) ds \\ & - \int_t^T U(s, t) Z(s) dW(s), \quad 0 \leq t \leq T. \end{aligned} \quad (2.77)$$

Furthermore, $\forall t \in [0, T]$

$$\mathbb{E} |Y(t)|_K^2 \leq 2M^2(T-t) \mathbb{E} \int_t^T |f(s)|_K^2 ds + 2M^2 \mathbb{E} |\xi|_K^2, \quad (2.78)$$

and

$$\begin{aligned} \mathbb{E} \int_t^T |Z(s)|_{L_2(H; K)}^2 ds \leq & 8M^2(T-t) \mathbb{E} \int_t^T |f(s)|_K^2 ds \\ & + 8M^2 \mathbb{E} |\xi|_K^2 ds, \end{aligned} \quad (2.79)$$

where

$$M := \sup_{0 \leq t \leq s \leq T} \|U(s, t)\|.$$

Proof. Uniqueness : Let (Y_1, Z_1) and (Y_2, Z_2) be two solutions of (2.77). Then

$$Y_1(t) - Y_2(t) = \int_t^T U(s, t) (Z_1(s) - Z_2(s)) dW(s).$$

Hence, by taking conditional expectation $\mathbb{E} [\cdot | \mathcal{F}_t]$ and using Proposition 2.22, we get that Y_1 and Y_2 are indistinguishable.

Existence : Define

$$Y(t) = \mathbb{E} [U(T, t) \xi + \int_t^T U(s, t) f(s) ds | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Therefore estimate (2.78) follows from this definition, Jensen's inequality and the assumptions in the lemma. By applying Theorem 1.2 to ξ and to $f(s)$, for each $s \in [0, T]$, we find that there exist two processes $z_1(s)$ and z_2 in $L^2_{\mathcal{F}}(0, T; L_2(H; K))$ such that

$$\begin{aligned} & \mathbb{E} [U(T, t) \xi + \int_t^T U(s, t) f(s) ds \mid \mathcal{F}_t] \\ &= \mathbb{E} [U(T, t) \xi + \int_t^T U(s, t) f(s) ds] \\ &+ \int_0^t U(T, t) z_2(r) dW(r) \\ &+ \int_0^t \int_t^s U(s, t) z_1(s)(r) dW(r) ds. \end{aligned} \quad (2.80)$$

Note that z_1 and z_2 are also the same processes given by (2.8) and (2.9). Subtract $U(T, t) \xi + \int_t^T U(s, t) f(s) ds$ from (2.80), then apply Fubini's theorem to obtain (2.77), with Z given by

$$Z(r) := U(T, r) z_2(r) + \int_r^T U(s, r) z_1(s)(r) ds, \quad 0 \leq r \leq T. \quad (2.81)$$

The estimate (2.79) follows easily from (2.81), (2.8) and (2.9). ■

Proposition 2.24 *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$ and $f : \Omega \times [0, T] \times L_2(H; K) \rightarrow K$ be a mapping satisfying (C1) and (C2). Then the following BSDE*

$$\begin{cases} -dY(t) = (A(t) Y(t) + f(t, Z(t))) dt - Z(t) dW(t), & 0 \leq t \leq T \\ Y(T) = \xi, \end{cases} \quad (2.82)$$

has a unique evolution solution (Y, Z) .

The proof of this proposition is achieved by approximation using Lemma 2.23, similar to that in [25, Proposition 2.5], when replacing the semigroups $\{e^{A(s-t)}, 0 \leq t \leq s \leq T\}$ by the evolution system $\{U(s, t), 0 \leq t \leq s \leq T\}$.

Using Proposition 2.24 we can construct a sequence of BSEEs in the evolution case, analogously to (2.12), with the evolution operator $U(s, t)$ replacing the semigroup $e^{A(s-t)}$. Then by using the same reasoning for proving Lemmas (2.6 , 2.7 , 2.8), one can easily get a version of each of these lemmas for the evolution case, as we did for proving Lemma 2.23, which is a generalisation of Lemma 2.2 to the evolution case. Consequently, in particular, we have the following version of Theorem 2.1 in the evolution case, by using the same idea of proof.

Theorem 2.25 *Assume that f and ξ satisfy (C1) and (C2). Then there exists a unique evolution solution to (2.75). Moreover,*

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y(t)|_K^2 < \infty ,$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T U(s, t) Z(s) dW(s) \right|_K^2 < \infty .$$

The proof of the estimates in the theorem is the same as for the semigroup case in Proposition 2.12.

The same regularity properties mentioned in Remark 2.13 also hold here for the solution (Y, Z) of (2.75) with the same argument.

Let us now give the following example to see how the solution in the theorem looks like.

Example 2.26 *Assume that $A(t)$, $0 \leq t \leq T$, in the theorem takes the form $A(t) = A + C(t)$, where $C : [0, T] \rightarrow L(K)$ is point measurable³ and essentially bounded. Consider the following BSEE*

$$\begin{cases} -dY(t) = ((A + C(t)) Y(t) + f(t, Y(t), Z(t))) dt - Z(t) dW(t) , \\ Y(T) = \xi . \end{cases} \quad (2.83)$$

³e.g. Borel measurable.

Then under the conditions in Theorem 2.25, the unique evolution solution (Y, Z) of (2.83) is given by

$$Y(t) = e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} C(s) Y(s) ds + \int_t^T e^{A(s-t)} f(s, Y(s), Z(s)) ds - \int_t^T e^{A(s-t)} Z(s) dW(s), \quad (2.84)$$

$0 \leq t \leq T$.

On the other hand, by looking at equation (2.84), we see that (Y, Z) is actually the mild solution of the following BSEE

$$\begin{cases} -dY(t) = (A Y(t) + \tilde{f}(t, Y(t), Z(t))) dt - Z(t) dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases}$$

which is given, with the help of Theorem 2.1, by

$$Y(t) = e^{A(T-t)} \xi + \int_t^T e^{A(s-t)} \tilde{f}(s, Y(s), Z(s)) ds - \int_t^T e^{A(s-t)} Z(s) dW(s), \quad 0 \leq t \leq T.$$

Here \tilde{f} is defined by

$$\tilde{f}(t, y, z) := f(t, y, z) + C(t) y.$$

In the rest of this section we shall discuss some types of solutions of the BSEE (2.75).

We shall assume that there exists a real separable Hilbert space \mathcal{D} , densely and continuously embedded in K such that $\mathcal{D} \subset \bigcap_{s \in [0, T]} \mathcal{D}(A^*(s))$, and $\forall s \in [0, T]$, $A^*(s) \in L(\mathcal{D}; K)$, i.e a bounded linear operator from \mathcal{D} to K . We require also the mapping $[0, T] \ni s \mapsto A^*(s) y \in K$ to be measurable for each $y \in K$.

In the semigroup case we take $\mathcal{D} = D(A^*)$ together with its graph norm.

We now introduce the following definitions of the types of solutions to the BSEE (2.75).

Definition 2.27 A $K \times L_2(H; K)$ - valued process (Y, Z) is called a strong solution of BSEE (2.75) if

- (i) $Y \in L^2_{\mathcal{F}}(0, T; K)$ and $Z \in L^2_{\mathcal{F}}(0, T; L_2(H; K))$,
- (ii) $Y(t) \in \mathcal{D}(A(t))$ a.s. for a.e. $0 \leq t \leq T$ and $A(\cdot) Y(\cdot) \in L^1([0, T]; K)$ a.s.
- (iii) For every $t \in [0, T]$ we have a.s.

$$Y(t) = \xi + \int_t^T (A(s)Y(s) + f(s, Y(s), Z(s))) ds - \int_t^T Z(s) dW(s). \quad (2.85)$$

Definition 2.28 A $K \times L_2(H; K)$ - valued process (Y, Z) is called a weak solution to BSEE (2.75) if

- (i) $Y \in L^2_{\mathcal{F}}(0, T; K)$ and $Z \in L^2_{\mathcal{F}}(0, T; L_2(H; K))$.
- (ii) For every $\rho \in \mathcal{D}$ we have a.s. $\langle Y(\cdot), A^*(\cdot)\rho \rangle_K \in L^1([0, T]; \mathbb{R})$.
- (iii) For every $t \in [0, T]$ and $\forall \rho \in \mathcal{D}$ we have a.s.

$$\begin{aligned} \langle Y(t), \rho \rangle_K &= \langle \xi, \rho \rangle_K + \int_t^T \langle Y(s), A^*(s)\rho \rangle_K ds \\ &+ \int_t^T \langle f(s, Y(s), Z(s)), \rho \rangle_K ds - \int_t^T \langle Z(s) dW(s), \rho \rangle_K. \end{aligned} \quad (2.86)$$

Definition 2.29 A $K \times L_2(H; K)$ - valued process (Y, Z) is called a weak evolution solution to BSEE (2.75) if

- (i) $Y \in L^2_{\mathcal{F}}(0, T; K)$ and $Z \in L^2_{\mathcal{F}}(0, T; L_2(H; K))$,
- (ii) For every $t \in [0, T]$ and $\forall \rho \in K$ we have a.s.

$$\begin{aligned} \langle Y(t), \rho \rangle_K &= \langle \xi, U^*(T, t)\rho \rangle_K \\ &+ \int_t^T \langle U^*(s, t)\rho, f(s, Y(s), Z(s)) \rangle_K ds \\ &- \int_t^T \langle U^*(s, t)\rho, Z(s) dW(s) \rangle_K. \end{aligned} \quad (2.87)$$

Proposition 2.30 A strong solution of (2.75) is a weak solution of (2.75).

Conversely, if a weak solution (Y, Z) of (2.75) satisfies condition (ii) of Definition 2.27, then (Y, Z) is a strong solution of (2.75).

Proof. The first part of the proposition is clear. The second part comes from the fact that

$$\int_t^T \langle Y(s), A^*(s) \rho \rangle_K ds = \int_t^T \langle A(s) Y(s), \rho \rangle_K ds,$$

for all $\rho \in \mathcal{D}$ and the fact that \mathcal{D} is dense in K . ■

Proposition 2.31 *An evolution solution of (2.75) is a weak evolution solution of (2.75) and vice versa.*

The idea of the proof of this proposition is the same as in Theorem 2.17.

As we saw in the proof of Theorem 2.17 the method of proving the relationships between some types of solutions is based on the commutativity of A and its semigroup $e^{A \cdot t}$. This is not the case, in general, when working with the evolution case. This makes it unclear, for example, to show that a strong solution of (2.75) is an evolution solution. For the forward SDEs driven by Wiener processes or even semimartingales, this difficulty can be overcome by putting some restrictions on the evolution operator U , namely to be of classes of weak backward adjoint operators (WBA) and weak forward adjoint evolution operators (WFA). See for example [27], [21] and references therein. See also [13] for the deterministic case.

Such an approach does not obviously work in the backward case, i.e. for BSEEs of type (2.75). In the next chapter we will put some restriction on the operators $A(t)$, $t \geq 0$, which allows us to deal with this equation directly and makes the solution rather more regular, although, we shall be working with a more general equations than (2.75).

Chapter 3

Backward Stochastic Partial Differential Equations in Infinite Dimensions

3.1 Introduction and Stochastic Integration with Respect to Martingales

We shall start this section by a brief introduction, without giving proofs, on stochastic integration in infinite dimensions with respect to Hilbert valued square integrable martingales. For convenience, we shall try to make our notation here very similar to those used in [35] and [34]. We refer the reader to these two references for further details; see also [53] for stochastic integration with respect to continuous square integrable martingales.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space, such that $\{\mathcal{F}_t\}_{t \geq 0}$ is right continuous.

Denote respectively by \mathcal{R} and \mathcal{P} the algebra generated by elements of $\Omega \times (0, T]$ of the form $F \times (t, s]$, where $F \in \mathcal{F}_t$ and $t, s \in [0, T]$, and the σ -algebra generated by \mathcal{R} . The space \mathcal{P} is called the *predictable* σ -algebra and its elements are called predictable sets. An H -valued process is said to be predictable if it is $\mathcal{P}/\mathcal{B}(H)$ measurable.

Denote by $\mathcal{M}_{[0,T]}^2(H)$ the vector space of cadlag square integrable martingales $\{M(t), 0 \leq t \leq T\}$, taking values in H , that is $\mathbb{E} |M(t)|_H^2 < \infty$ for every $t \in [0, T]$. It is a separable Hilbert space with respect to the inner product $(M, N) \mapsto \mathbb{E} \langle M(T), N(T) \rangle_H$, if we agree to identify \mathbb{P} -equivalence classes. Thus two elements M and N of $\mathcal{M}_{[0,T]}^2(H)$ are *orthogonal* if $\mathbb{E} \langle M(T), N(T) \rangle_H = 0$. This property is weaker than being $\mathbb{E} \langle M(u), N(u) \rangle_H = 0$, for all $[0, T]$ -valued stopping times u ; in which case we say that M and N are *strongly orthogonal*. There is also another type of orthogonality, the so-called *very strong orthogonality*: M and N are said to be very strong orthogonal (V.S.O.) if $\mathbb{E} (M(u) \otimes N(u)) = 0$, for all $[0, T]$ -valued stopping times u . Note that $\langle M(t), N(t) \rangle_H = \text{tr}(M(t) \otimes N(t))$. Therefore we have the following implications: very strong orthogonality \Rightarrow strong orthogonality \Rightarrow orthogonality. Here \otimes denotes the tensor product. If $x \in H$, we shall denote $x \otimes x = x^{\otimes 2}$.

Let $\mathcal{M}_{[0,T]}^{2,c}(H)$ denote that the subspace of $\mathcal{M}_{[0,T]}^2(H)$, consisting of those continuous square integrable martingales. It is also a Hilbert subspace of $\mathcal{M}_{[0,T]}^2(H)$. See [34] for the proof.

Let us recall the definition of *Doléans measure* associated with $|M|_H^2$. Define $d_{|M|_H^2}$ on elements $A = F \times (t, s]$ of \mathcal{R} by $d_{|M|_H^2}(A) := \mathbb{E} [1_F(|M(s)|_H^2 - |M(t)|_H^2)]$. This function can be extended in a unique way to a measure on \mathcal{P} , we call this measure the Doléans measure associated with $|M|_H^2$. Denote it by α_M . Analogously, we associate also the Doléans function of $M \otimes M$, that is $d_{M \otimes M}(F \times (t, s]) = \mathbb{E} [1_F(M^{\otimes 2}(s) - M^{\otimes 2}(t))] = \mathbb{E} [1_F(M(s) - M(t))^{\otimes 2}] \in H \hat{\otimes}_1 H \subset H \hat{\otimes}_2 H$, for every predictable rectangle $(t, s] \times F$. Let μ_M denote the extension of $d_{M \otimes M}$ to an $H \hat{\otimes}_1 H$ -valued σ -additive measure on \mathcal{P} . See [35] and [34] for more details. It is easily seen from these definitions that $\alpha_M = \text{tr} \mu_M$. Here the space $H \hat{\otimes}_1 H$ is the completed nuclear tensor product, that is the completion of $H \otimes H$ for the nuclear norm. Recall that the linear form trace, denoted here by tr , is defined as the unique continuous extension to $H \hat{\otimes}_1 H$ of the mapping $x \otimes y \mapsto \langle x, y \rangle_H$. An element b of $H \hat{\otimes}_1 H$ is said

to be *symmetric* if $\langle b, x \otimes y \rangle_{H \hat{\otimes}_2 H} = \langle b, y \otimes x \rangle_{H \hat{\otimes}_2 H}$, for every $x, y \in H$. It is said to be *positive* if it is symmetric and $\langle b, x \otimes x \rangle_{H \hat{\otimes}_2 H} \geq 0$, for every $x \in H$.

For a square integrable martingale M we write $\langle M, M \rangle$ (or for short $\langle M \rangle$) for the increasing Meyer process associated with the Doléans measure of the submartingale $|M|_H^2$, that is the unique predictable cadlag increasing process such that $|M|_H^2 - \langle M \rangle$ is a martingale. It exists since $|M|_H^2$ is a submartingale.

The proof of the following proposition can be found in [35, Theorem 14.3.1, P. 167].

Proposition 3.1 (1) *There is one predictable $H \hat{\otimes}_1 H$ -valued process \mathcal{Q}_M , defined up to α_M -equivalence such that for every $G \in \mathcal{P}$*

$$\mu_M(G) = \int_G \mathcal{Q}_M d\alpha_M.$$

Moreover, \mathcal{Q}_M takes its value in the set of positive symmetric elements of $H \hat{\otimes}_1 H$ and

$$\text{tr } \mathcal{Q}_M(\omega, t) = 1, \quad \alpha_M \text{ a.e.}$$

(2) *The $H \hat{\otimes}_1 H$ -valued process*

$$\langle\langle M \rangle\rangle_t := \int_{[0,t]} \mathcal{Q}_M d\langle M \rangle$$

has finite variation, is predictable, admits μ_M as its Doléans measure, and is such that $M^{\otimes 2} - \langle\langle M \rangle\rangle$ is a martingale.

Thus M and N are V.S.O. if and only if $\langle\langle M, N \rangle\rangle = 0$.

Note that if, for example, we have a 2-dimensional Brownian motion $B = (B^1, B^2)$, where B^1 and B^2 are two independent Brownian motions in \mathbb{R} , then $\langle\langle B \rangle\rangle_t = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} =: t I_2$. Hence $\langle B \rangle_t = 2t$ and $\mathcal{Q}_B = \frac{1}{2} I_2$. Also, μ_B is the product measure $(l \otimes \mathbb{P}) I_2$ and $\alpha_B = (2l \otimes \mathbb{P})$, where l is the Lebesgue measure on $([0, T], \mathcal{B}([0, T]))$.

Recall that any elements of $H \hat{\otimes}_1 H$ can be identified with an element of $L_1(H; H)$ ($L_1(H)$ for short), the space of nuclear operators on H , and vice versa. Therefore we can associate with \mathcal{Q}_M the $L_1(H)$ -valued process $\tilde{\mathcal{Q}}_M$ by

$$\langle \tilde{\mathcal{Q}}_M(h), g \rangle_H = \langle \mathcal{Q}_M, h \otimes g \rangle_{H \hat{\otimes}_2 H}, \quad (h, g) \in H \times H.$$

Let $L^*(H; K; \mathcal{P}, M)$ be the space of processes Φ , the values of which are (possibly non-continuous) linear operators from H into K , with the following properties:

- (i) the domain of $\Phi(\omega, t)$ contains $\tilde{\mathcal{Q}}_M^{1/2}(\omega, t)(H)$ for every (ω, t) ,
- (ii) for every $h \in H$, the K -valued process $\Phi \circ \tilde{\mathcal{Q}}_M^{1/2}(h)$ is predictable ,
- (iii) for every $(\omega, t) \in \Omega \times (0, T]$, $\Phi(\omega, t) \circ \tilde{\mathcal{Q}}_M^{1/2}(\omega, t)$ is a Hilbert-Schmidt operator and

$$\int_{\Omega \times (0, T]} |\Phi \circ \tilde{\mathcal{Q}}_M^{1/2}|_{L_2(H; K)}^2 d\alpha_M < \infty.$$

This space is complete with respect to the scalar product $(X, Y) \mapsto \int_{\Omega \times (0, T]} \text{tr} (X \circ \tilde{\mathcal{Q}}_M \circ Y^*) d\alpha_M$; cf. [34, Proposition 22.2, P.142].

Denote by $\mathcal{E}(L(H; K))$ the space of \mathcal{R} -simple processes and $\Lambda^2(H; K; \mathcal{P}, M)$ the closure of $\mathcal{E}(L(H; K))$ in $L^*(H; K; \mathcal{P}, M)$. It is therefore a Hilbert subspace of $L^*(H; K; \mathcal{P}, M)$.

For a simple Φ of the form

$$\Phi = \sum_{i=1}^n 1_{F_i \times (r_i, s_i]} u_i, \quad u_i \in L(H; K), \quad F_i \in \mathcal{F}_{r_i},$$

we define

$$\int_{(0, t]} \Phi dM = \sum_{i=1}^n 1_{F_i \times (r_i, s_i]} (u_i(M(s_i \wedge t)) - u_i(M(r_i \wedge t))), \quad t \in [0, T].$$

We thus have defined an isometric linear mapping from $\mathcal{E}(L(H; K))$ into $\mathcal{M}_{[0, T]}^2(K)$, $\Phi \mapsto \int \Phi dM$. Extend this mapping to $\Lambda^2(H; K; \mathcal{P}, M)$. We call the image $\int \Phi dM$ of Φ in $\mathcal{M}_{[0, T]}^2(K)$, by this mapping, the *stochastic integral of Φ with respect to M* .

We can also make use of this definition to define the stochastic integral of the following predictable H -valued process. Let ϕ be a predictable H -valued process such that

$$\mathbb{E} \int_0^T |\Phi(s)|_H^2 ds < \infty.$$

Define $\tilde{\Phi}$ by $\tilde{\Phi}(s)h := \langle \Phi(s), h \rangle_H$, $h \in H$. Then $\tilde{\Phi} \in \Lambda^2(H; H; \mathcal{P}, M)$ (or $\Lambda^2(H; \mathcal{P}, M)$ for short). We thus write $\int_{(0,t]} \langle \Phi, dM \rangle_H$ to mean $\int_{(0,t]} \tilde{\Phi} dM$. In fact

$$\int_{(0,t]} \langle \Phi, dM \rangle_H = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{(0,t]} \langle \Phi(s), h_i \rangle_H dM_i(s),$$

for some o.n. basis $\{h_i\}_{i \geq 1}$ in H , where $M_i := \langle M, h_i \rangle_H$.

Let us now present some properties of this integral. Let $M \in M_{[0,T]}^2(H)$ and $\Phi \in \Lambda^2(H; K; \mathcal{P}, M)$. Then $N = \int \Phi dM$ satisfies the following items:

1. $\alpha_N = \text{tr} (\Phi \circ \tilde{Q}_M \circ \Phi^*) \alpha_M$,
2. $\tilde{Q}_N = (\text{tr} (\Phi \circ \tilde{Q}_M \circ \Phi^*))^{-1} \Phi \circ \tilde{Q}_M \circ \Phi^*$,
3. $\mu_N = (\Phi \circ \tilde{Q}_M \circ \Phi^*) \alpha_M = (\Phi \otimes \Phi) \mu_M$,
4. $\langle N \rangle_t = \int_{(0,t]} \text{tr} (\Phi \circ \tilde{Q}_M \circ \Phi^*) ds$,
5. $\langle\langle N \rangle\rangle_t = \int_{(0,t]} (\Phi \circ \tilde{Q}_M \circ \Phi^*) d\langle\langle M \rangle\rangle$.

For the proof see [34, Proposition 22.7, P.146].

One can check easily the above properties for the case when M is a \mathcal{Q} -Wiener process W ; in which case $\langle W \rangle_t = t \operatorname{tr} \mathcal{Q}$ and $\langle \langle W \rangle \rangle_t = t \mathcal{Q}$. Thus $\tilde{\mathcal{Q}}_W = \frac{\mathcal{Q}}{\operatorname{tr} \mathcal{Q}}$ and α_W is the product measure on $(l \otimes \mathbb{P}) \operatorname{tr} \mathcal{Q}$.

The following representation theorem is due to [38]; cf. also [34, E. 8, P. 160].

Theorem 3.2 *Let $M \in \mathcal{M}_{[0,T]}^2(H)$, $\mathcal{H}_1 := \{ \int X \, dM : X \in \Lambda^2(H; K; \mathcal{P}, M) \} \subset \mathcal{M}_{[0,T]}^2(K)$. Let \mathcal{H}_2 be the orthogonal complement of \mathcal{H}_1 in $\mathcal{M}_{[0,T]}^2(K)$. Then every element of \mathcal{H}_2 is V.S.O. to every element of in \mathcal{H}_1 . In particular, every $L \in \mathcal{M}_{[0,T]}^2(K)$ can be written uniquely as*

$$L = \int X \, dM + N, \quad X \in \Lambda^2(H; K; \mathcal{P}, M), \quad N \in \mathcal{H}_2.$$

We now extend this result to allow M to be a cylindrical Wiener process W on H with respect to $\{\mathcal{F}_t, 0 \leq t \leq T\}$. For simplicity we shall let the spaces H and K be the same.

Let $L \in \mathcal{M}_{[0,T]}^2(H)$. Define

$$L_1(t) := \mathbb{E} [L(t) \mid \mathcal{F}_t(W)].$$

Then for $s \leq t$ we have

$$\mathbb{E} [L_1(t) \mid \mathcal{F}_s(W)] = \mathbb{E} [\mathbb{E} [L(t) \mid \mathcal{F}_s] \mid \mathcal{F}_s(W)] = \mathbb{E} [L(s) \mid \mathcal{F}_s(W)],$$

i.e. L_1 is a martingale with respect to $\{\mathcal{F}_t(W), 0 \leq t \leq T\}$.

By using Theorem 1.2, we obtain a unique $\Phi \in L_{\mathcal{F}(W)}^2(0, T; L_2(H))$ ($\subseteq L_{\mathcal{F}}^2(0, T; L_2(H))$) such that

$$L_1(t) = \int_0^t \Phi(s) \, dW(s), \quad t \in [0, T].$$

It is known that such Φ has an $\mathcal{F}_*(W)$ predictable modification. We shall consider this modification. Hence Φ is predictable and lies in $\Lambda^2(H; \mathcal{P}, W)$,

which now is simply the space of $L_2(H)$ -valued predictable processes ψ such that $\mathbb{E} \int_0^T |\psi(s)|_{L_2(H)}^2 ds < \infty$. Thus L_1 is martingale with respect to $\{\mathcal{F}_t, 0 \leq t \leq T\}$. Define N such that $N(t) = L(t) - L_1(t)$ for all $t \in [0, T]$. Then it is easily seen that $N \in \mathcal{M}_{[0,T]}^2(H)$.

We now show that N is V.S.O. to W , in the sense that for any h and $g \in H$ the following holds

$$\mathbb{E} \langle W(u), h \rangle_H \langle N(u), g \rangle_H = 0,$$

for all $[0, T]$ -stopping times u .

In fact

$$\mathbb{E} \langle N_1(u) \otimes N_2(u), h \otimes g \rangle_{H \otimes_2 H} = \mathbb{E} \langle N_1(u), h \rangle_H \langle N_2(u), g \rangle_H,$$

for N_1 and $N_2 \in \mathcal{M}_{[0,T]}^2(H)$.

To see now that W and N are V.S.O., note that

$$\begin{aligned} & \mathbb{E} \langle W(u), h \rangle_H \langle N(u), g \rangle_H \\ &= \mathbb{E} [\mathbb{E} [\langle W(u), h \rangle_H \langle N(u), g \rangle_H \mid \mathcal{F}_u(W)]] \\ &= \mathbb{E} [\langle W(u), h \rangle_H \mathbb{E} [\langle N(u), g \rangle_H \mid \mathcal{F}_u(W)]] = 0, \end{aligned}$$

since

$$\begin{aligned} & \mathbb{E} [\langle N(u), g \rangle_H \mid \mathcal{F}_u(W)] \\ &= \mathbb{E} [\mathbb{E} [\langle N(T), g \rangle_H \mid \mathcal{F}_u] \mid \mathcal{F}_u(W)] \\ &= \mathbb{E} [\langle N(T), g \rangle_H \mid \mathcal{F}_u(W)] \\ &= \mathbb{E} [\mathbb{E} [\langle N(T), g \rangle_H \mid \mathcal{F}_T(W)] \mid \mathcal{F}_u(W)] \\ &= \mathbb{E} [\langle \mathbb{E} [L(T) \mid \mathcal{F}_T(W)] - L_1(T), g \rangle_H \mid \mathcal{F}_u(W)] = 0, \end{aligned}$$

by the definition of L_1 . Thus we have proved the following theorem.

Theorem 3.3 *Let W be a cylindrical Wiener process on H with respect to $\{\mathcal{F}_t, 0 \leq t \leq T\}$. Suppose that $L \in \mathcal{M}_{[0,T]}^2(H)$. Then L can be written uniquely as*

$$L = \int \Phi dW + N,$$

with $\Phi \in \Lambda^2(H; \mathcal{P}; W)$, for some $N \in \mathcal{M}_{[0,T]}^2(H)$, which is V.S.O. to W .

Notice that in this theorem N is also V.S.O. to $\int \Phi dW$. In fact one can show clearly by using the same argument used above that $\mathbb{E} (L_1(u) \otimes N(u)) = 0$, for all such stopping times. From this we get that

$$\mathbb{E} |L(T)|_H^2 = \mathbb{E} \int_0^T |\Phi(s)|_{L_2(H)}^2 ds + \mathbb{E} |N(T)|_H^2.$$

Note that if $L \in \mathcal{M}^{2,c}(H)$, then N above has a continuous modification. In such a case, we shall consider this continuous modification.

Applications of these two former theorems will be established in the rest of this chapter.

3.2 General Backward Stochastic Differential Equations

In Chapter 1 we studied those BSDEs driven by either a cylindrical Wiener process or by a genuine Wiener process taking values in H . The solution was required to be adapted to the filtration generated by this driving Wiener process, i.e. the Wiener filtration. The question arising now is whether we are able to deal with such sort of BSDEs with a given arbitrary filtration, not necessary the Wiener filtration. For example a filtration generated by a martingale $M \in \mathcal{M}_{[0,T]}^2(H)$. Another example could be the filtration generated by two independent cylindrical Wiener processes W_1 and W_2 on H . If the terminal value ξ of the concerned BSDEs is measurable with respect to a $\mathcal{F}_T(W_1)$ and independent of $\mathcal{F}_T(W_2)$, then the solution (Y, Z) in general can only be adapted to $\{\mathcal{F}_t(W_1) \vee \mathcal{F}_t(W_2), 0 \leq t \leq T\}$, since the generator $f(t, 0, 0)$ is $\mathcal{F}_t(W_1) \vee \mathcal{F}_t(W_2)$ adapted, for all $0 \leq t \leq T$.

By working with a general filtration we are able to study more equations than those studied in Chapter 1 and Chapter 2 as well. The following equation, which we shall call a *general backward stochastic differential equation* (or GBSDE for short), is an example where the filtration is arbitrary.

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t)) dt - Z(t) dM(t) - dN(t), & 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases} \quad (3.1)$$

where f , ξ and M satisfy the following conditions. Let $\gamma > 0$ be fixed.

- (D1) f is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \mathcal{B}(L_2^{\tilde{Q}^M}(H)) / \mathcal{B}(H)$ -measurable ;
- (D2) $\exists k > 0$ such that $\forall y, y' \in H, \forall z, z' \in L_2^{\tilde{Q}^M}(H)$

$$|f(t, y, z) - f(t, y', z')|_H^2 \leq k (|y - y'|_H^2 + |z - z'|_{L_2^{\tilde{Q}^M}(H)}^2),$$

uniformly in (t, ω) ;

- (D3) $M \in \mathcal{M}_{[0, T]}^2(H)$, cadlag and $\langle\langle M \rangle\rangle_t = \int_0^t b(s) b(s)^* dc_s$, for some adapted continuous and increasing \mathbb{R}_+ -valued process $\{c_s, s \geq 0\}$ such that $c_0 = 0$, and an $L_2(H)$ -valued predictable process b . In other words, M is absolutely continuous with respect to c .
- (D4) $\mathbb{E} \int_0^T e^{\gamma c_s} |f(s, 0, 0)|_H^2 dc_s < \infty$.
- (D5) $\mathbb{E} (e^{\gamma c_T} |\xi|_H^2) < \infty$.

It follows from (D3) that, for all t ,

$$\langle M \rangle_t = \int_0^t |b(s)|_{L_2(H)}^2 dc_s$$

and

$$\tilde{Q}_M(t) = \frac{b(t) b(t)^*}{|b(t)|_{L_2(H)}^2}.$$

Thus condition (D2) becomes

$$|f(t, y, z) - f(t, y', z')|_H^2 \leq k (|y - y'|_H^2 + |(z - z') b(t)|_{L_2(H)}^2). \quad (3.2)$$

Note that the equation (3.1) is actually, a BSDE driven by the martingale M .

Let us now introduce the following spaces.

$\hat{L}_{\mathcal{F}}^2(0, T; H) := \{ \phi : [0, T] \times \Omega \rightarrow H, \text{ predictable and } \mathbb{E} \int_0^T e^{\gamma t} |\phi(t)|_H^2 dt < \infty \}.$

$\mathcal{S}^2(H) := \{ \phi : [0, T] \times \Omega \rightarrow H, \text{ cadlag, adapted and } \mathbb{E} \sup_{0 \leq t \leq T} e^{\gamma t} |\phi(t)|_H^2 < \infty \}.$

$\hat{\Lambda}^2(H; \mathcal{P}, M) := \{ \phi : [0, T] \times \Omega \rightarrow H, \text{ predictable and } \mathbb{E} \int_0^T e^{\gamma t} |\phi(t) b(t)|_H^2 dt < \infty \}.$

$\hat{\mathcal{M}}_{[0, T]}^2(H) := \{ N \in \mathcal{M}_{[0, T]}^2(H), \mathbb{E} \int_0^T e^{\gamma t} d \langle N \rangle_t < \infty \}.$

$\mathcal{B}_1^2(H) := \mathcal{S}^2(H) \times \hat{\Lambda}^2(H; \mathcal{P}, M).$

$\mathcal{B}_2^2(H) := \hat{L}_{\mathcal{F}}^2(0, T; H) \times \hat{\Lambda}^2(H; \mathcal{P}, M).$

Then $\mathcal{B}_1^2(H)$ is a separable Banach space equipped with the norm

$$|(\phi_1, \phi_2)|_{\mathcal{B}_1^2(H)}^2 := \mathbb{E} \sup_{0 \leq t \leq T} e^{\gamma t} |\phi_1(t)|_H^2 + \mathbb{E} \int_0^T e^{\gamma t} |\phi_2(t) b(t)|_H^2 dt.$$

Also $\mathcal{B}_2^2(H)$ is a separable Hilbert space with the norm

$$|(\phi_1, \phi_2)|_{\mathcal{B}_2^2(H)}^2 := \mathbb{E} \int_0^T e^{\gamma t} |\phi_1(t)|_H^2 dt + \mathbb{E} \int_0^T e^{\gamma t} |\phi_2(t) b(t)|_H^2 dt.$$

A solution to (3.1) is a triple $(Y, Z, N) \in \mathcal{B}_2^2(H) \times \hat{\mathcal{M}}_{[0, T]}^2(H)$ such that for all $t \in [0, T]$, we have a.s.

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T Z(s) dM(s) - \int_t^T dN(s), \quad (3.3)$$

and $\langle\langle N, M \rangle\rangle = 0$, i.e. M and N are V.S.O.

The first main theorem of this section is the following.

Theorem 3.4 *Suppose that (D1)–(D5) hold with a parameter γ large enough. There exists a unique solution $(Y, Z, N) \in \mathcal{B}_2^2(H) \times \hat{\mathcal{M}}_{[0, T]}^2(H)$ of (3.1).*

Moreover, $(Y, Z, N) \in \mathcal{B}_1^2(H) \times \hat{\mathcal{M}}_{[0, T]}^2(H)$.

A similar result in finite dimensions can be found in [16], under slightly weaker conditions than ours here.

Proof of Theorem 3.4 . Let $(y, z) \in \mathcal{B}_2^2(H)$. Note that for all $t \in [0, T]$ and for all $\beta > 0$, we have

$$\begin{aligned} & \left(\int_t^T |f(s, y(s), z(s))|_H \, dc_s \right)^2 \\ &= \left(\int_t^T e^{(-\beta/2) c_s} e^{(\beta/2) c_s} |f(s, y(s), z(s))|_H \, dc_s \right)^2 \\ &\leq \frac{1}{\beta} e^{-\beta c_t} \int_t^T e^{\beta c_s} |f(s, y(s), z(s))|_H^2 \, dc_s . \end{aligned} \quad (3.4)$$

Therefore, by putting $\beta := \frac{\gamma}{2}$ in this inequality (3.4) and using Fubini's theorem, we get

$$\begin{aligned} & \int_0^T e^{\gamma c_t} \left(\int_t^T |f(s, y(s), z(s))|_H \, dc_s \right)^2 \, dc_t \\ &\leq \frac{4}{\gamma^2} \int_0^T e^{\gamma c_t} |f(t, y(t), z(t))|_H^2 \, dc_t . \end{aligned} \quad (3.5)$$

Thus by using (3.2)

$$\begin{aligned} & \int_0^T e^{\gamma c_t} \left(\int_t^T |f(s, y(s), z(s))|_H \, dc_s \right)^2 \, dc_t \\ &\leq \frac{8k}{\gamma^2} \int_0^T e^{\gamma c_s} |y(s)|_H^2 \, dc_s + \frac{8k}{\gamma^2} \int_0^T e^{\gamma c_s} |z(s) b(s)|_{L_2(H)}^2 \, dc_s \\ &+ \frac{8}{\gamma^2} \int_0^T e^{\gamma c_s} |f(s, 0, 0)|_H^2 \, dc_s . \end{aligned} \quad (3.6)$$

Consider now the following process

$$Y(t) = \mathbb{E} \left[\xi + \int_t^T f(s, y(s), z(s)) \, dc_s \mid \mathcal{F}_t \right] , \quad 0 \leq t \leq T. \quad (3.7)$$

From Cauchy-Schwartz inequality and (3.6), we derive that

$$\begin{aligned}
& \mathbb{E} \int_0^T e^{\gamma c_t} |Y(t)|_H^2 dc_t \\
& \leq 2 \mathbb{E} \int_0^T \mathbb{E} [e^{\gamma c_t} |\xi|_H^2 | \mathcal{F}_t] dc_t \\
& + 2 \mathbb{E} \int_0^T \mathbb{E} [e^{\gamma c_t} (\int_t^T |f(s, y(s), z(s))|_H dc_s)^2 | \mathcal{F}_t] dc_t \\
& \leq \frac{2}{\gamma} \mathbb{E} (e^{\gamma c_T} |\xi|_H^2) + \frac{16k}{\gamma^2} \mathbb{E} \int_0^T e^{\gamma c_s} |y(s)|_H^2 dc_s \\
& + \frac{16k}{\gamma^2} \mathbb{E} \int_0^T e^{\gamma c_s} |z(s) b(s)|_{L_2(H)}^2 dc_s \\
& + \frac{16}{\gamma^2} \mathbb{E} \int_0^T e^{\gamma c_s} |f(s, 0, 0)|_H^2 dc_s .
\end{aligned} \tag{3.8}$$

This together with assumptions (D4) and (D5) gives $Y \in \hat{L}_{\mathcal{F}}^2(0, T; H)$.

Next we show that $Y \in \mathcal{S}^2(H)$.

By using (3.7), (3.4) (with $\beta = \gamma$), Doob's inequality and assumptions (3.2), (D4) and (D5), we obtain

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} e^{\gamma c_t} |Y(t)|_H^2 \\
& = \mathbb{E} \sup_{0 \leq t \leq T} e^{\gamma c_t} | \mathbb{E} [\xi + \int_t^T f(s, y(s), z(s)) dc_s | \mathcal{F}_t] |_H^2 \\
& \leq 2 \mathbb{E} \sup_{0 \leq t \leq T} (\mathbb{E} [e^{\gamma c_T} |\xi|_H^2 \\
& + e^{\gamma c_t} (\int_t^T |f(s, y(s), z(s))|_H dc_s)^2 | \mathcal{F}_t]) \\
& \leq 8 \mathbb{E} (e^{\gamma c_T} |\xi|_H^2) + \frac{16k}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |y(s)|_H^2 dc_s \\
& + \frac{16k}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |z(s) b(s)|_{L_2(H)}^2 dc_s \\
& + \frac{16}{\gamma} \int_0^T e^{\gamma c_s} |f(s, 0, 0)|_H^2 dc_s < \infty .
\end{aligned} \tag{3.9}$$

Define the mapping Φ on $\mathcal{B}_2^2(H)$, which maps $(y, z) \in \mathcal{B}_2^2(H)$ to the

process (Y, Z) , where Y is given by (3.7) and Z is given uniquely, by using Theorem 3.2, as follows

$$\begin{aligned} & \mathbb{E} \left[\xi + \int_0^T f(s, y(s), z(s)) \, dc_s \mid \mathcal{F}_t \right] \\ &= Y(0) + \int_0^t Z(s) \, dM(s) + N(t), \quad 0 \leq t \leq T, \end{aligned}$$

for some $N \in \mathcal{M}_{[0,T]}^2(H)$ verifying $\langle\langle M, N \rangle\rangle = 0$.

We now show that $Z \in \hat{\Lambda}^2(H; \mathcal{P}, M)$; hence $(Y, Z) \in \mathcal{B}_2^2(H)$.

Recall first that, for each $t \in [0, T]$,

$$\begin{aligned} & \mathbb{E} \left| \int_t^T (Z(s) \, dM(s) + dN(s)) \right|_H^2 \\ &= \mathbb{E} \int_t^T |Z(s) \, b(s)|_{L_2(H)}^2 \, dc_s + \mathbb{E} \int_t^T d \langle N \rangle_s, \end{aligned}$$

for all $t \in [0, T]$. By using the integration by parts and this equality, we get

$$\begin{aligned} & \mathbb{E} \int_0^T e^{\gamma c_s} |Z(s) \, b(s)|_{L_2(H)}^2 \, dc_s + \mathbb{E} \int_0^T e^{\gamma c_s} d \langle N \rangle_s \\ &= \mathbb{E} \left| \xi - Y(0) + \int_0^T f(s, y(s), z(s)) \, dc_s \right|_H^2 \\ &+ \gamma \mathbb{E} \int_0^T e^{\gamma c_s} \left| \xi - Y(s) + \int_s^T f(r, y(r), z(r)) \, dc_r \right|_H^2 \, dc_s \\ &\leq 3 \left(\mathbb{E} |\xi|_H^2 + \mathbb{E} |Y(0)|_H^2 \right) + 3 \mathbb{E} \left(\int_0^T |f(s, y(s), z(s))|_H \, dc_s \right)^2 \\ &+ 3 \gamma \mathbb{E} \int_0^T e^{\gamma c_s} |\xi|_H^2 \, dc_s + 3 \gamma \mathbb{E} \int_0^T e^{\gamma c_s} |Y(s)|_H^2 \, dc_s \\ &+ 3 \gamma \mathbb{E} \int_0^T e^{\gamma c_s} \left(\int_s^T |f(r, y(r), z(r))|_H \, dc_r \right)^2 \, dc_s. \end{aligned} \tag{3.10}$$

But

$$\begin{aligned}
\mathbb{E} (|\xi|_H^2 + |Y(0)|_H^2) &\leq \mathbb{E} (e^{\gamma c_T} |\xi|_H^2) + \mathbb{E} \sup_{0 \leq t \leq T} e^{\gamma c_t} |Y(t)|_H^2 \\
&\leq 9 \mathbb{E} (e^{\gamma c_T} |\xi|_H^2) + \frac{16k}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |y(s)|_H^2 dc_s \\
&\quad + \frac{16k}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |z(s) b(s)|_{L_2(H)}^2 dc_s \\
&\quad + \frac{16}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |f(s, 0, 0)|_H^2 dc_s, \tag{3.11}
\end{aligned}$$

by using (3.9). Substitute (3.11) in (3.10), use (3.4) (with $\beta = \gamma$), (3.2), (3.8) and (3.6) to get that

$$\begin{aligned}
&\mathbb{E} \int_0^T e^{\gamma c_s} |Z(s) b(s)|_{L_2(H)}^2 dc_s + \mathbb{E} \int_0^T e^{\gamma c_s} d \langle N \rangle_s \\
&\leq 36 \mathbb{E} (e^{\gamma c_T} |\xi|_H^2) + \frac{126k}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |y(s)|_H^2 dc_s \\
&\quad + \frac{126k}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |z(s) b(s)|_{L_2(H)}^2 dc_s \\
&\quad + \frac{126}{\gamma} \mathbb{E} \int_0^T e^{\gamma c_s} |f(s, 0, 0)|_H^2 dc_s. \tag{3.12}
\end{aligned}$$

Hence $Z \in \hat{\Lambda}^2(H; \mathcal{P}, M)$ and $N \in \hat{\mathcal{M}}_{[0, T]}^2(H)$. Thus we have shown that the mapping Φ maps $\hat{\mathcal{B}}_2^2(H)$ into itself. Therefore (Y, Z, N) is the solution of the following GBSDE

$$Y(t) = \xi + \int_t^T f(s, y(s), z(s)) dc_s - \int_t^T Z(s) dM(s) - \int_t^T dN(s),$$

$0 \leq t \leq T$.

We now show that Φ becomes actually a contraction on $\hat{\mathcal{B}}_2^2(H)$.

Let (y_1, z_1) and (y_2, z_2) be two elements of $\hat{\mathcal{B}}_2^2(H)$ and let (Y_1, Z_1) and (Y_2, Z_2) denote respectively their images in $\hat{\mathcal{B}}_2^2(H)$ under Φ . Thus (Y_i, Z_i, N_i) is the solution of the GBSDE with the generator $f(t, y_i(t), z_i(t))$ and terminal value equals to ξ , for $i = 1, 2$.

Denote $\delta Y = Y_1 - Y_2$, $\delta Z = Z_1 - Z_2$, $\delta N = N_1 - N_2$. Then $\delta N \in \hat{\mathcal{B}}_2^2(H)$ and $\int_0^\cdot \delta Z(s) dM(s) \in \hat{\Lambda}^2(H; \mathcal{P}, M)$. We have for all $t \in [0, T]$

$$\begin{aligned} \delta Y(t) &= \int_t^T (f(s, y_1(s), z_1(s)) - f(s, y_2(s), z_2(s))) dc_s \\ &\quad - \int_t^T \delta Z(s) dM(s) - \int_t^T d\delta N(s). \end{aligned}$$

By doing the same way as we did for (3.8) and for (3.12), we derive that

$$\begin{aligned} &\mathbb{E} \int_0^T e^{\gamma c_t} |\delta Y(t)|_H^2 dc_t + \mathbb{E} \int_0^T e^{\gamma c_t} |\delta z(t) b(t)|_{L_2(H)}^2 dc_t \\ &\leq \frac{C}{\gamma} \left(\mathbb{E} \int_0^T e^{\gamma c_t} |\delta y(t)|_H^2 dc_t \right. \\ &\quad \left. + \mathbb{E} \int_0^T e^{\gamma c_t} |\delta z(t) b(t)|_{L_2(H)}^2 dc_t \right), \end{aligned} \quad (3.13)$$

for some positive constant $C > 0$. Therefore, by choosing $\gamma > C$, we find that Φ is a contraction mapping on $\hat{\mathcal{B}}_2^2(H)$. Hence it has a unique fixed point $(Y, Z) \in \hat{\mathcal{B}}_2^2(H)$. It is immediately seen from the definition of Φ that (Y, Z, N) is the unique solution of the GBSDE (3.1). Here the martingale $N \in \mathcal{M}_{[0, T]}^2(H)$ is given, with the help of Theorem 3.2, by

$$\begin{aligned} &\mathbb{E} \left[\xi + \int_0^T f(s, Y(s), Z(s)) dc_s \mid \mathcal{F}_t \right] \\ &= Y(0) + \int_0^t Z(s) dM(s) + N(t), \quad 0 \leq t \leq T. \end{aligned}$$

The fact that this solution Y lies in $\mathcal{S}^2(H)$ comes from (3.9). ■

Remark 3.5 (i) As we saw in the proof, the process Y solving the GBSDE (3.1) is only known to be right continuous; so it may develop a jump. The continuity does not play a big deal here.

(ii) The results of Theorem 3.4 can be taken somehow as a generalisation of some other equations, namely those called reflected BSDEs; see e.g. [17].

Since the solution of these equations takes values usually in \mathbb{R} , to make use of our result here, one should first re-write Theorem 3.4 for the case when the martingale M lies in the space H , while Y lies in \mathbb{R} . This is however straightforward.

On the other hand, our results in this theorem have also proved to be very useful in the study of approximation of BSDEs, as it is shown in [7]. Precisely, consider the BSDE (1.16) appearing in Chapter 1, with W being a genuine Wiener process. By taking a martingale approximation of this W (see [7] for definition), we obtain a sequence of equations, all of which are of the type (3.1). It was shown in [7] that this sequence of solutions actually converge to the solution we started with. This result is done in [7] for the finite dimensional case.

Warning : From here on, for simplicity, we shall let the constant γ be zero in the definition of the spaces $\hat{L}_{\mathcal{F}}^2(0, T; H)$, $\mathcal{S}^2(H)$, $\hat{\mathcal{M}}_{[0, T]}^2(H)$. Thus we have

$$\begin{aligned}\hat{L}_{\mathcal{F}}^2(0, T; H) &:= \{ \phi : [0, T] \times \Omega \rightarrow H, \text{ predictable and } \mathbb{E} \int_0^T |\phi(t)|_H^2 dt < \infty \} . \\ \mathcal{S}^2(H) &:= \{ \phi : [0, T] \times \Omega \rightarrow H, \text{ cadlag, adapted and } \mathbb{E} \sup_{0 \leq t \leq T} |\phi(t)|_H^2 < \infty \} ; \\ \hat{\Lambda}^2(H; \mathcal{P}, W) &:= \Lambda^2(H; \mathcal{P}, W) ; \\ \hat{\mathcal{M}}_{[0, T]}^2(H) &:= \mathcal{M}_{[0, T]}^2(H) ; \\ \mathcal{B}_1^2(H) &:= \mathcal{S}^2(H) \times \Lambda^2(H; \mathcal{P}, W) ; \\ \mathcal{B}_2^2(H) &:= \hat{L}_{\mathcal{F}}^2(0, T; H) \times \Lambda^2(H; \mathcal{P}, W) .\end{aligned}$$

We let the process c be such that $c_t = t$, all t , since we shall be working with a cylindrical Wiener process, as a driving noise of our GBSDE. Particularly, consider the following GBSDE:

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t)) dt - Z(t) dW(t) - dN(t) , & 0 \leq t \leq T, \\ Y(T) = \xi , \end{cases} \quad (3.14)$$

where W here is a cylindrical Wiener process on H .

A first seen at this equation excludes it from the scope of (3.1), for being W cylindrical. We shall therefore give a separate study for this equation for

a matter of completeness. The reader, however, will see that the proof is not too different from that offered to the equation (3.1) in Theorem 3.4.

Let us first make the following changes to the previous conditions.

- (E1) f is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \mathcal{B}(L_2(H))/\mathcal{B}(H)$ -measurable and $\mathbb{E} \int_0^T |f(t, 0, 0)|_H^2 dt < \infty$;
- (E2) $\exists k > 0$ such that $\forall y, y' \in H, \forall z, z' \in L_2(H)$

$$|f(t, y, z) - f(t, y', z')|_H^2 \leq k (|y - y'|_H^2 + |z - z'|_{L_2(H)}^2)$$

uniformly in (t, ω) ;

- (E3) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$.

A solution to (3.14) is a triple $(Y, Z, N) \in \mathcal{B}_2^2(H) \times \mathcal{M}_{[0,T]}^2(H)$ such that for all $t \in [0, T]$ the following equality holds *a.s.*

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s) - \int_t^T dN(s), \quad (3.15)$$

where N is V.S.O. to W .

The next result is the second main result of this section. It is an application of Theorem 3.3.

Theorem 3.6 *If (E1)–(E3) holds, then there exists a unique solution $(Y, Z, N) \in \mathcal{B}_2^2(H) \times \mathcal{M}_{[0,T]}^2(H)$ to (3.14).*

Moreover, $Y \in \mathcal{S}^2(H)$.

The proof of this theorem is done best by using the fixed point theorem, similar but much simpler than that is for the previous theorem.

Proof. Define the mapping Φ on $\mathcal{B}_1^2(H)$ as follows: $\mathcal{B}_1^2(H) \ni (y, z) \mapsto (Y, Z)$, where $Y(t) := \mathbb{E} [\xi + \int_t^T f(s, y(s), z(s)) ds \mid \mathcal{F}_t]$, $0 \leq t \leq T$, and Z is given by using Theorem 3.3 as

$$\mathbb{E} [\xi + \int_0^T f(s, y(s), z(s)) ds \mid \mathcal{F}_t] = Y(0) + \int_0^t Z(s) dW(s) + N(t), \quad (3.16)$$

$0 \leq t \leq T$, where N is a cadlag local martingale in H , that is V.S.O. to W .

By using Doob's inequality and assumptions (E1)–(E3), we derive that

$$\begin{aligned} |Y|_{\mathcal{S}^2(H)}^2 &\leq 8 \mathbb{E} [|\xi|_H^2 + T \int_0^T |f(s, y(s), z(s))|_H^2 ds] \\ &< \infty. \end{aligned} \quad (3.17)$$

Hence $Y \in \mathcal{S}^2(H)$.

On the other hand, we derive

$$\begin{aligned} &\mathbb{E} [\int_0^T |Z(s)|_{L_2(H)}^2 ds + \langle N \rangle_T] \\ &= \mathbb{E} | \int_0^T Z(s) dW(s) + N(T) |_H^2 \\ &= \mathbb{E} | -Y(0) + \xi + \int_0^T f(s, y(s), z(s)) ds |_H^2 \\ &\leq 9 (\mathbb{E} |\xi|_H^2 + T \int_0^T |f(s, y(s), z(s))|_{L_2(H)}^2 ds) \\ &< \infty, \end{aligned} \quad (3.18)$$

by using assumptions (E1)–(E3). Therefore $N \in \mathcal{M}_{[0,T]}^2(H)$ and Φ maps $\mathcal{B}_1^2(H)$ into itself.

Take two elements (y_i, z_i) of $\mathcal{B}_1^2(H)$, $i = 1, 2$, with the corresponding ones (Y_i, Z_i, N_i) in $\mathcal{B}_1^2(H) \times \mathcal{M}_{[0,T]}^2(H)$, $i = 1, 2$, via the mapping Φ . Thus $(\delta Y, \delta Z, \delta N) := (Y_1 - Y_2, Z_1 - Z_2, N_1 - N_2)$ is the solution of the following GBSDE

$$\begin{aligned} \delta Y(t) &= \int_t^T (f(s, y_1(s), z_1(s)) - f(s, y_2(s), z_2(s))) ds \\ &\quad - \int_t^T \delta Z(s) dW(s) - \int_t^T d\delta N(s). \end{aligned}$$

As we did for the two estimates (3.17) and (3.18), with the help of assumption (E2), we find that if $T \leq 1$, then

$$|(\delta Y, \delta Z)|_{\mathcal{B}_1^2(H)}^2 \leq C T |(\delta y, \delta z)|_{\mathcal{B}_1^2(H)}^2.$$

for some positive constant C . Thus, by choosing T so that $C T < 1$, we conclude that Φ a contraction.

Thus it follows that Φ has a unique fixed point (Y, Z) such that $(Y, Z, N) \in \mathcal{B}_1^2(H) \times \mathcal{M}_{[0, T]}^2(H)$ is the solution of (3.14), where N is got uniquely by using Theorem 3.3, as we did in the proof of Theorem 3.4.

The rest of the proof is the same as in the preceding proof. Thus it follows that there exists a unique triple $(Y, Z, N) \in \mathcal{B}_1^2(H) \times \mathcal{M}_{[0, T]}^2(H)$, which solves (3.14).

The fact that Y is predictable is seen from its definition in (3.16). Hence $(Y, Z, N) \in \mathcal{B}_2^2(H) \times \mathcal{M}_{[0, T]}^2(H)$.

Finally, the reader may find the cure to our restriction on T , which we made earlier in the proof, by taking a finite number of suitable partitions to the interval $[0, T]$. The proof is complete. ■

Note that the solution process Y in the theorem is also not continuous in t , in general. It becomes continuous if, for example, the right hand side of (3.15) is continuous. This is the case when the martingale N has a continuous version, e.g. when the filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is the σ -algebra generated by two independent cylindrical Wiener processes on H . This will be our restriction on the filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$ in the next section.

3.3 General Backward Stochastic Partial Differential Equations

This section is devoted to studying the following equation:

$$\begin{cases} -dY(t) = (A(t)Y(t) + f(t, Y(t), Z(t)) + g(t, Y(t)))dt \\ \quad - Z(t)dW(t) - dN(t), \quad 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases} \quad (3.19)$$

or equivalently, for all $t \in [0, T]$,

$$\begin{aligned} Y(t) = & \xi + \int_t^T (A(s) Y(s) + f(s, Y(s), Z(s)) + g(s, Y(s))) ds \\ & - \int_t^T Z(s) dW(s) - \int_t^T dN(s) . \end{aligned} \quad (3.20)$$

A special case of these equations were already studied in Section 2.4, where we showed the existence and uniqueness of an evolution solution. In that sense the solution satisfied another type of equation; see (2.76) and Example 2.26.

In this section we shall allow the operator $A(t)$ to be random as well. We shall see below how we can solve equation (3.19) directly, using Galerkin's approximation method for monotone operators; see Theorem 3.7 below. In fact we shall show that the solution process Y takes its values in a space V that is continuously embedded in the space H , where the process Y usually would evolve.

Such equations have proved to be very useful in the study of the adjoint equation of an optimal control of a quasilinear stochastic heat equation, see [37], in which the Wiener filtration is considered. See also [4].

The setting here is as follows.

Let (V, H, V') be a *rigged* Hilbert space ([62] or [53]), that is V is a separable Hilbert space embedded continuously and densely in H . Hence by identifying H with its dual, we obtain the following continuous and dense two inclusions: $V \subseteq H \subseteq V'$, where V' is the dual space of V . In fact this is seen as follows. For every $h \in H$, there corresponds $\bar{h} : V \rightarrow \mathbb{R}$, defined by $\bar{h}(v) := \langle h, v \rangle_H, v \in V$, which is a linear continuous functional since $|\bar{h}(v)| \leq |h|_H |v|_H \leq \text{const } |h|_H |v|_V$. I.e. $\bar{h} \in V'$. Moreover, the mapping $h \mapsto \bar{h}$ from H to V' is linear, injective and continuous. The injectivity of this mapping comes from the definition of \bar{h} above and the density of $V \subseteq H$. Thus we may and we will identify \bar{h} with h . We then

have $|h|_{V'} \leq \text{const } |h|_H$, $\forall h \in H$. Thus the following embedding $H \subseteq V'$ has a sense and, moreover, it is continuous and dense.

Denote by $[\cdot, \cdot]$ the duality between V and V' .

Let us record the following properties of this bilinear form:

- (i) $|[v, x]| \leq \text{const } |v|_V \cdot |x|_{V'}$, $\forall v \in V$ and $x \in V'$;
- (ii) $[v, x] = \langle v, x \rangle_H$ if $x \in H$.

Now we set the following assumptions on the variables of the above equation.

- (F1) f is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \mathcal{B}(L_2(H))/\mathcal{B}(H)$ -measurable and $\mathbb{E} \int_0^T |f(t, 0, 0)|_H^2 dt < \infty$;
- (F2) $\exists k > 0$ such that $\forall y, y' \in H, \forall z, z' \in L_2(H)$

$$|f(t, y, z) - f(t, y', z')|_H^2 \leq k (|y - y'|_H^2 + |z - z'|_{L_2(H)}^2)$$

uniformly in (t, ω) ;

- (F3) $A(t, \omega)$ is a linear operator on H , \mathcal{P} -measurable, belongs to $L(V; V')$ uniformly in (t, ω) and satisfies the following *coercivity* condition:

$$2 [A(t, \omega) y, y] - \lambda |y|_H^2 \leq -\alpha |y|_V^2 \quad \text{a.e. } t \in [0, T], \quad \text{a.s. } \forall y \in V,$$

for some $\alpha, \lambda > 0$.

- (F4) g is $\mathcal{P} \otimes \mathcal{B}(H)/\mathcal{B}(V')$ -measurable and $\mathbb{E} \int_0^T |g(t, 0)|_{V'}^2 dt < \infty$;
- (F5)¹ $|g(t, y) - g(t, y')|_{V'}^2 \leq k |y - y'|_H^2$, $\forall y, y' \in H$, uniformly in (t, ω) .
- (F6) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$.

¹for simplicity, we use here the same Lipschitz constant k as in (F2)

We assume also that every martingale with respect $\{\mathcal{F}_t, 0 \leq t \leq T\}$ has a continuous version. An example of this is the filtration generated by two cylindrical Wiener processes on H .

Our aim is to prove the following theorem.

Theorem 3.7 *Assume that (F1)–(F6) hold. Then there exists a unique triple $(Y, Z, N) \in \hat{L}_{\mathcal{F}}^2(0, T; V) \times \Lambda^2(H; \mathcal{P}, W) \times \mathcal{M}_{[0, T]}^{2, c}(H)$ such that equation (3.20) holds a.s. for all $t \in [0, T]$ and N is V.S.O. to W .*

In the case of this theorem we say that (3.19) has a unique solution.

Before going into the proof of this theorem let us record here a remark that, after we completed the thesis, we realised that actually the mapping g above can be taken to depend on the variable z as well, in a similar way as the mapping f does in (F2). Particularly, assumptions (F4) and (F5) are changed in an obvious way to accept such a change.

In this case, unless some one would like to weaken some of the Lipschitz conditions on the mapping g , in which case Lemma 3.8 below remains more general, there is no need to have the mapping f in the equation (3.19). The proof to this case goes in a parallel way to the proof we give below for Theorem 3.7.

We start the proof of Theorem 3.7 by proving firstly a crucial lemma, in which we shall consider the following simple version of the equation (3.19).

$$\begin{aligned} Y(t) = & \xi + \int_t^T (A(s) Y(s) + f(s) + g(s)) ds \\ & - \int_t^T Z(s) dW(s) - \int_t^T dN(s), \quad 0 \leq t \leq T, \end{aligned} \quad (3.21)$$

Lemma 3.8 *Suppose that $f \in L_{\mathcal{F}}^2(0, T; H)$, $g \in L_{\mathcal{F}}^2(0, T; V')$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$. Then (3.21) has a unique solution $(Y, Z, N) \in \hat{L}_{\mathcal{F}}^2(0, T; V) \times \Lambda^2(H; \mathcal{P}, W) \times \mathcal{M}_{[0, T]}^{2, c}(H)$.*

We shall use the method of *Galerkin's* finite dimensional approximation, following [39]. This method is an extension to the stochastic case of the that

used by J. Lions [30] for the deterministic case. Among those who used this method of approximation in the stochastic case are e.g. Bensoussan [3], [4], Rozovskiĭ [53] and Pardoux [39]. See also the work of Gyöngy and Krylov, e.g. in [23]. Other useful discussion on this can also be found in [32].

Proof of Lemma 3.8.

Uniqueness : Let (Y, Z, N) and (Y', Z', N') be two solutions of (3.21). By using Itô's formula and assumption (F3), we get

$$\begin{aligned} & \mathbb{E} |Y(t) - Y'(t)|_H^2 + \mathbb{E} \int_t^T |Z(s) - Z'(s)|_{L_2(H)}^2 ds \\ & + \mathbb{E} \int_t^T d < N - N' >_s + \alpha \mathbb{E} \int_t^T |Y(s) - Y'(s)|_V^2 ds \\ & \leq \lambda \mathbb{E} \int_t^T |Y(s) - Y'(s)|_H^2 ds, \quad 0 \leq t \leq T. \end{aligned} \quad (3.22)$$

In particular,

$$\mathbb{E} |Y(t) - Y'(t)|_H^2 \leq \lambda \mathbb{E} \int_t^T |Y(s) - Y'(s)|_H^2 ds, \quad 0 \leq t \leq T.$$

Thus by using Gronwall's inequality and the continuity of Y and Y' , we conclude that $Y(t) = Y'(t) \quad \forall t \in [0, T] \quad a.s.$ This together with (3.22) gives the uniqueness of Z and N .

Existence : Let $\{e_i\}_{i \geq 1}$ be an o.n. base of H . Since V is dense in H , we can and we will assume that $e_i \in V$ for each $i \geq 1$. Let $H_n := \text{span}(e_1, e_2, \dots, e_n)$, $n \geq 1$.

Consider the following system of equations in $H_n \cong \mathbb{R}^n$:

$$\begin{aligned} Y_n^i(t) &= \langle e_i, \xi \rangle_H + \int_t^T [\langle e_i, A(s) \sum_{j=1}^n Y_n^j(s) \cdot e_j + g(s) \rangle_H] ds \\ &+ \int_t^T \langle e_i, f(s) \rangle_H ds \\ &- \int_t^T Z_n^i(s) dW^n(s) - \int_t^T dN_n^i(s), \end{aligned} \quad (3.23)$$

$i = 1, 2, \dots, n$, where W^n is as in section 1.2. A solution to this system is a triple $(Y_n^i, Z_n^i, N_n^i) \in \hat{L}_{\mathcal{F}}^2(0, T; \mathbb{R}) \times \Lambda^2(H_n; \mathbb{R}; \mathcal{P}, W) \times \mathcal{M}_{[0, T]}^{2, c}(\mathbb{R})$ and N_n^i is V.S.O. to W^n , $\forall n \geq 1$.

Define $Y_n(t) := \sum_{i=1}^n Y_n^i(t) e_i$, $Z_n(t) := \sum_{i=1}^n Z_n^i(t)(\cdot) e_i$, $N_n(t) := \sum_{i=1}^n N_n^i(t) e_i$, $0 \leq t \leq T$. Then we have the the following GBSDE:

$$\begin{aligned} Y_n(t) = & \pi_n \xi + \int_t^T (\Pi_n A(s) Y_n(s) + \Pi_n g(s)) ds \\ & + \int_t^T \pi_n f(s) ds - \int_t^T Z_n(s) dW^n(s) - \int_t^T dN_n(s), \end{aligned} \quad (3.24)$$

$0 \leq t \leq T$, where $\Pi_n : V' \rightarrow H_n$ and $\pi_n : H \rightarrow H_n$ are the operators of orthogonal projections. Note that the second integral in (3.24) is finite since

$$\mathbb{E} \int_0^T |A(s) Y_n(s)|_{V'}^2 ds < \text{const.} \mathbb{E} \int_0^T |Y_n(s)|_V^2 ds.$$

It is also easily seen that equation (3.24) verifies the conditions in Theorem 3.6 (for fixed n). Thereby (3.24) attains a unique solution $(Y_n, Z_n, N_n) \in \mathcal{B}_2^2(H_n) \times \mathcal{M}_{[0, T]}^{2, c}(H_n)$.

By using Itô's formula, we see that

$$\begin{aligned} \mathbb{E} |Y_n(t)|_H^2 = & \mathbb{E} |\pi_n \xi|_H^2 + 2 \mathbb{E} \int_t^T [Y_n(s), \Pi_n A(s) Y_n(s) + \Pi_n g(s)] ds \\ & + 2 \mathbb{E} \int_t^T \langle Y_n(s), \pi_n f(s) \rangle_H ds \\ & - \mathbb{E} \int_t^T |\tilde{Z}_n(s)|_{L_2(H)}^2 ds - \mathbb{E} \int_t^T d \langle N_n \rangle_s, \end{aligned} \quad (3.25)$$

where $\tilde{Z}(s) := Z(s) \pi_n$. Thus by using (F3), we get

$$\begin{aligned} & \mathbb{E} |Y_n(t)|_H^2 + \alpha \mathbb{E} \int_t^T |Y_n(s)|_V^2 ds \\ & + \mathbb{E} \int_t^T |\tilde{Z}_n(s)|_{L_2(H)}^2 ds + \mathbb{E} \int_t^T d \langle N_n \rangle_s \\ & \leq \mathbb{E} |\xi|_H^2 + (\lambda + 1) \mathbb{E} \int_t^T |Y_n(s)|_H^2 ds + \mathbb{E} \int_t^T |f(s)|_H^2 ds. \end{aligned} \quad (3.26)$$

Hence by using Gronwall's inequality

$$\mathbb{E} |Y_n(t)|_H^2 \leq e^{(\lambda+1)T} (\mathbb{E} |\xi|_H^2 + \mathbb{E} \int_0^T |f(s)|_H^2 ds) ,$$

which yields that

$$\mathbb{E} \int_0^T |Y_n(t)|_H^2 \leq C_1 , \quad (3.27)$$

where $C_1 := T e^{(\lambda+1)T} (\mathbb{E} |\xi|_H^2 + \mathbb{E} \int_0^T |f(s)|_H^2 ds)$. Therefore, by using this inequality (3.27) with (3.26), we obtain the following estimates:

$$\sup_{n \geq 1} \mathbb{E} \int_0^T |Y_n(s)|_H^2 ds < \infty , \quad (3.28)$$

$$\sup_{n \geq 1} \mathbb{E} \int_0^T |Y_n(s)|_V^2 ds < \infty , \quad (3.29)$$

$$\sup_{n \geq 1} \mathbb{E} \int_0^T |\tilde{Z}_n(s)|_{L_2(H)}^2 ds < \infty , \quad (3.30)$$

and

$$\sup_{n \geq 1} \mathbb{E} |N_n(T)|_H^2 < \infty . \quad (3.31)$$

It follows from these estimates that, for some subsequence $\{n_k, k \geq 1\}$, $(Y_{n_k}, \tilde{Z}_{n_k}, N_{n_k})$ converges weakly in $\hat{L}_{\mathcal{F}}^2(0, T; V) \times \Lambda^2(H; \mathcal{P}, W) \times \mathcal{M}_{[0, T]}^{2, c}(H)$, as $k \rightarrow \infty$. Call their limit (Y, Z, N) .

It remains to show that (Y, Z, N) is a solution to (3.21).

Let $\phi \in L_0^{2,1}([0, T]; \mathbb{R})$. Define $\phi_i := \phi e_i$. Hence $\phi_i \in L_0^{2,1}([0, T]; H)$.

Multiply (3.23) by ϕ and then apply Itô's formula to find that

$$\begin{aligned} & \langle \xi, \phi_i(T) \rangle_H + \int_0^T [A(s) Y_n(s) + g(s), \phi_i(s)] ds \\ & + \int_0^T \langle f(s), \phi_i(s) \rangle_H ds \\ & - \int_0^T \langle \phi_i(s), \tilde{Z}_n(s) dW(s) \rangle_H - \int_0^T \langle \phi_i(s), dN_n(s) \rangle_H \\ & = \int_0^T \langle Y_n(s), \dot{\phi}_i(s) \rangle_H ds . \end{aligned} \quad (3.32)$$

The integral $\int_0^T [A(s) Y_n(s) , \phi_i(s)] ds$ exists in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ by using property (i) of the bilinear functional $[\cdot, \cdot]$ and estimate (3.29).

Now replace n by n_k in (3.32) and pass to the weak limit in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, as $k \rightarrow \infty$, to get

$$\begin{aligned} & \langle \xi , \phi_i(T) \rangle_H + \int_0^T [A(s) Y(s) + g(s) , \phi_i(s)] ds \\ & + \int_0^T \langle f(s) , \phi_i(s) \rangle_H ds \\ & - \int_0^T \langle \phi_i(s) , Z(s) dW(s) \rangle_H - \int_0^T \langle \phi_i(s) , dN(s) \rangle_H \\ & = \int_0^T \langle Y(s) , \dot{\phi}_i(s) \rangle_H ds . \end{aligned} \quad (3.33)$$

To clarify this equality, note that the mapping

$$\Psi : \mathcal{M}^{2,c}(H) \rightarrow L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) , \quad M \mapsto \int_0^T \langle \phi_i(s) , dM(s) \rangle_H$$

is continuous since

$$\begin{aligned} \mathbb{E} |\Psi(M)|_{\mathbb{R}}^2 & \leq \mathbb{E} \int_0^T |\phi_i(s)|_H^2 ds \langle M \rangle_s \\ & \leq C_2 \mathbb{E} |M(T)|_H^2 , \end{aligned}$$

for some constant $C_2 > 0$. Also the following mapping

$$\begin{aligned} \Phi : \Lambda^2(H; \mathcal{P}, W) & \rightarrow L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) , \\ R & \mapsto \int_0^T \langle \phi_i(s) , R(s) dW(s) \rangle_H \end{aligned}$$

is continuous since

$$\begin{aligned} \mathbb{E} |\Phi(R)|_{\mathbb{R}}^2 & = \mathbb{E} \int_0^T |\phi(s)|_{\mathbb{R}}^2 ds \langle (\int_0^\cdot R(k) dW(k) , e_i \rangle_H) \rangle_s \\ & \leq C_3 \mathbb{E} | \int_0^T R(s) dW(s) , e_i \rangle_H |_{\mathbb{R}}^2 \\ & \leq C'_3 \mathbb{E} \int_0^T |R(s)|_{L_2(H)}^2 ds , \end{aligned}$$

by Cauchy-Schwartz inequality, where C_3 and C'_3 are some positive constants.

By using property (i) of $[\cdot, \cdot]$, it is immediately seen that the following mapping φ :

$$Y \mapsto \int_0^T [A(s) Y(s), \phi_i(s)] ds ,$$

is also continuous and linear from $\hat{L}^2_{\mathcal{F}}(0, T; V)$ to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

These three mappings Ψ and Φ and φ are linear as well. Therefore Ψ , Φ and φ are continuous with respect to the weak topologies. This implies (3.33).

For being (3.33) holds for every $i \geq 1$, we find that

$$\begin{aligned} & \langle \xi, v \rangle_H \phi(T) + \int_0^T [A(s) Y(s) + g(s), v] \phi(s) ds \\ & + \int_0^T \langle f(s), v \rangle_H \phi(s) ds \\ & - \int_0^T \phi(s) \langle v, Z(s) dW(s) \rangle_H - \int_0^T \phi(s) \langle v, dN(s) \rangle_H \\ & = \int_0^T \dot{\phi}(s) \langle Y(s), v \rangle_H ds , \end{aligned} \quad (3.34)$$

for every $v \in V$.

Let $t \in (0, T)$. Choosing

$$\phi_m(s) := \begin{cases} 1 & \text{if } s \geq t + \frac{1}{2m} , \\ \frac{1}{2} - m(t - s) & \text{if } t - \frac{1}{2m} < s < t + \frac{1}{2m} , \\ 0 & \text{if } s \leq t - \frac{1}{2m} , \end{cases} \quad (3.35)$$

for $m \geq 1$, in (3.34), we derive

$$\begin{aligned} & \langle \xi, v \rangle_H + \int_0^T [A(s) Y(s) + g(s), v] \phi_m(s) ds \\ & + \int_0^T \langle f(s), v \rangle_H \phi_m(s) ds \\ & - \int_0^T \phi_m(s) \langle v, Z(s) dW(s) \rangle_H - \int_0^T \phi_m(s) \langle v, dN(s) \rangle_H \\ & = m \int_{t - \frac{1}{2m}}^{t + \frac{1}{2m}} \langle Y(s), v \rangle_H ds , \end{aligned} \quad (3.36)$$

for all $m \geq 1$. Thus by taking the limit as $m \rightarrow \infty$ in (3.36), it follows that for almost all $t \in [0, T]$ we have

$$\begin{aligned} & \langle \xi, v \rangle_H + \int_t^T [A(s) Y(s) + g(s), v] ds + \int_t^T \langle f(s), v \rangle_H ds \\ & - \int_t^T \langle v, Z(s) dW(s) \rangle_H - \int_t^T \langle v, dN(s) \rangle_H \\ & = \langle Y(t), v \rangle_H . \end{aligned} \quad (3.37)$$

By using the separability of V we conclude from (3.37) that, for *a.e.* $t \in [0, T]$,

$$\begin{aligned} Y(t) &= \xi + \int_t^T (A(s) Y(s) + g(s) + f(s)) ds \\ &- \int_t^T Z(s) dW(s) - \int_t^T dN(s) . \end{aligned} \quad (3.38)$$

The process defined by the right hand side of this equation (3.38) has a continuous modification, we therefore define Y to be this process.

It remains to show that N is V.S.O. to W .

Let u be an $[0, T]$ -valued stopping time. We have to show that

$$\mathbb{E} (W^h(u) \cdot N^g(u)) = 0 , \quad (3.39)$$

for every h and $g \in H$, where $W^h := \langle W, h \rangle_H$ and $N^g := \langle N, g \rangle_H$.

Note that we have, for each $k \geq 1$,

$$\mathbb{E} W^{n_k, h}(u) \cdot N_{n_k}^g(u) = 0 ,$$

where $W^{n_k, h} := \langle W^{n_k}, h \rangle_H$. This implies that

$$\mathbb{E} W^{m, h}(u) \cdot N_{n_k}^g(u) = 0 ,$$

for every $m \leq n_k$, since

$$W^{m,h} = \int \langle \pi_m^{n_k} h, dW^{n_k} \rangle_H ,$$

where $\pi_m^{n_k} := \pi_m \circ \pi_{n_k}$.

Thus

$$\mathbb{E} W^{m,h}(T) \cdot N_{n_k}^g(u) = \mathbb{E} W^{m,h}(u) \cdot N_{n_k}^g(u) = 0 , \quad (3.40)$$

for all $m \leq n_k$.

On the other hand, since N_{n_k} converges weakly to N in $\mathcal{M}_{[0,T]}^{2,c}(H)$ as $k \rightarrow \infty$, then $N_{n_k}^g$ converges weakly to N^g in $\mathcal{M}_{[0,T]}^{2,c}(\mathbb{R})$ as $k \rightarrow \infty$. This implies, by using the Optional stopping theorem, that $N_{n_k}^g(u \wedge \cdot)$ converges weakly to $N^g(u \wedge \cdot)$ in $\mathcal{M}_{[0,T]}^{2,c}(\mathbb{R})$ as $k \rightarrow \infty$. Indeed, if $M \in \mathcal{M}_{[0,T]}^{2,c}(\mathbb{R})$, hence $M^u := M(u \wedge \cdot) \in \mathcal{M}_{[0,T]}^{2,c}(\mathbb{R})$ and so

$$\begin{aligned} \mathbb{E} N_{n_k}^g(u \wedge T) \cdot M(T) &= \mathbb{E} N_{n_k}^g(T) \cdot M^u(T) \\ &\rightarrow \mathbb{E} N^g(T) \cdot M^u(T) \\ &= \mathbb{E} N^g(u) \cdot M(T) . \end{aligned}$$

as $k \rightarrow \infty$, by using the weak convergence of $N_{n_k}^g$.

In particular,

$$\mathbb{E} N_{n_k}^g(u) \cdot W^{m,h}(T) \rightarrow \mathbb{E} N^g(u) \cdot W^{m,h}(T) ,$$

as $k \rightarrow \infty$. Now let $m \rightarrow \infty$ and use the strong (hence the weak) convergence of $W^{m,h}$ to get

$$\mathbb{E} N^g(u) \cdot W^{m,h}(T) \rightarrow \mathbb{E} N^g(u) \cdot W^h(T) .$$

This together with (3.40) implies that

$$\mathbb{E} N^g(u) \cdot W^h(T) = \mathbb{E} N^g(u) \cdot W^h(T) = 0 ,$$

which gives (3.39).

Thereby (Y, Z, N) is a solution to (3.21). ■

In the next step we let f depend on the variables t and Z but not on Y . In particular, we have

$$\begin{aligned} Y(t) = & \xi + \int_t^T (A(s) Y(s) + f(s, Z(s)) + g(s)) ds \\ & - \int_t^T Z(s) dW(s) - \int_t^T dN(s), \quad 0 \leq t \leq T, \end{aligned} \quad (3.41)$$

Lemma 3.9 *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ and let $g \in L^2_{\mathcal{F}}(0, T; V')$. Assume that $f : \Omega \times [0, T] \times L_2(H) \rightarrow H$ satisfying (F1) and (F2).*

Then there exists a unique solution (Y, Z, N) to (3.41) in $\hat{L}^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, W) \times \mathcal{M}^{2,c}(H)$.

Proof. The proof of uniqueness of solutions is achieved by using the same argument as in Lemma 3.8.

We now show the existence of solutions to (3.41).

The proof is done by approximation using Lemma 3.8. Let $Z_0 \equiv 0$. Consider the following equation

$$\begin{aligned} Y_n(t) = & \xi + \int_t^T (A(s) Y_n(s) + f(s, Z_{n-1}(s)) + g(s)) ds \\ & - \int_t^T Z_n(s) dW(s) - \int_t^T dN_n(s), \end{aligned} \quad (3.42)$$

for $0 \leq t \leq T$ and $n \geq 1$.

Then according to Lemma 3.8 this GBSPDE has a unique solution $(Y_n, Z_n, N_n) \in \hat{L}^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, W) \times \mathcal{M}^{2,c}(H)$, for each $n \geq 1$.

Apply Itô's formula to (3.42) and use (F2) and (F3) to obtain that

$$\begin{aligned}
& \mathbb{E} |Y_{n+1}(t) - Y_n(t)|_H^2 + \mathbb{E} \int_t^T |Z_{n+1}(s) - Z_n(s)|_{L_2(H)}^2 ds \\
& + \mathbb{E} \int_t^T d \langle N_{n+1} - N_n \rangle_s \\
& \leq (\lambda + 2k) \mathbb{E} \int_t^T |Y_{n+1}(s) - Y_n(s)|_H^2 ds \\
& + \frac{1}{2} \mathbb{E} \int_t^T |Z_n(s) - Z_{n-1}(s)|_{L_2(H)}^2 ds \\
& - \alpha \mathbb{E} \int_t^T |Y_{n+1}(s) - Y_n(s)|_V^2 ds .
\end{aligned} \tag{3.43}$$

By multiplying both sides of (3.43) by $e^{(\lambda+2k)t}$ and integrating $\int_0^T \cdot dt$, we get

$$\begin{aligned}
& \mathbb{E} \int_0^T |Y_{n+1}(t) - Y_n(t)|_H^2 dt \\
& + \int_0^T e^{(\lambda+2k)t} (\mathbb{E} \int_t^T |Z_{n+1}(s) - Z_n(s)|_{L_2(H)}^2 ds) dt \\
& + \int_0^T e^{(\lambda+2k)t} (\mathbb{E} \int_t^T d \langle N_{n+1} - N_n \rangle_s) dt \\
& \leq \frac{1}{2} \int_0^T e^{(\lambda+2k)t} (\mathbb{E} \int_t^T |Z_n(s) - Z_{n-1}(s)|_{L_2(H)}^2 ds) dt .
\end{aligned} \tag{3.44}$$

In particular, we have

$$\begin{aligned}
& \int_0^T e^{(\lambda+2k)t} (\mathbb{E} \int_t^T |Z_{n+1}(s) - Z_n(s)|_{L_2(H)}^2 ds) dt \\
& \leq \frac{1}{2} \int_0^T e^{(\lambda+2k)t} (\mathbb{E} \int_t^T |Z_n(s) - Z_{n-1}(s)|_{L_2(H)}^2 ds) dt .
\end{aligned} \tag{3.45}$$

But this inequality reads after iteration as

$$\begin{aligned}
& \int_0^T e^{(\lambda+2k)t} \left(\mathbb{E} \int_t^T |Z_{n+1}(s) - Z_n(s)|_{L_2(H)}^2 ds \right) dt \\
& \leq \left(\frac{1}{2} \right)^n \frac{1}{\lambda + 2k} e^{(\lambda+2k)T} \mathbb{E} \int_0^T |Z_1(s)|_{L_2(H)}^2 ds =: \left(\frac{1}{2} \right)^n C_5 . \quad (3.46)
\end{aligned}$$

Thus, by using (3.44), we get

$$\mathbb{E} \int_0^T |Y_{n+1}(t) - Y_n(t)|_H^2 dt \leq \left(\frac{1}{2} \right)^n C_5 . \quad (3.47)$$

On the other hand, inequalities (3.43) and (3.47) yield, after iterating in n , that

$$\begin{aligned}
& \mathbb{E} \int_0^T |Z_{n+1}(s) - Z_n(s)|_{L_2(H)}^2 ds \\
& \leq \left(\frac{1}{2} \right)^n \left[n(\lambda + 2k) C_5 + \mathbb{E} \int_0^T |Z_1(s)|_{L_2(H)}^2 ds \right] . \quad (3.48)
\end{aligned}$$

Moreover from (3.43), (3.47) and (3.48) we get

$$\begin{aligned}
& \mathbb{E} \int_0^T |Y_{n+1}(s) - Y_n(s)|_V^2 ds \leq \\
& \frac{1}{\alpha} \left(\frac{1}{2} \right)^n \left[n(\lambda + 2k) C_5 + \mathbb{E} \int_0^T |Z_1(s)|_{L_2(H)}^2 ds \right] , \quad (3.49)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} | (N_{n+1} - N_n)(T) |_H^2 \leq \\
& \left(\frac{1}{2} \right)^n \left[n(\lambda + 2k) C_5 + \mathbb{E} \int_0^T |Z_1(s)|_{L_2(H)}^2 ds \right] . \quad (3.50)
\end{aligned}$$

Therefore, from (3.49), (3.48) and (3.50), we conclude that the sequences $\{Y_n\}$, $\{Z_n\}$ and $\{N_n\}$, where $n \geq 1$, are Cauchy sequences in $\hat{L}_{\mathcal{F}}^2(0, T; V)$, $\Lambda^2(H; \mathcal{P}, W)$ and $\mathcal{M}^{2,c}(H)$, respectively. Call their limit (Y, Z, N) .

From the strong orthogonality between N_n and W , for each $n \geq 1$, and the weak convergence of N_n to N as $n \rightarrow \infty$, we deduce that N is V.S.O. to W , as done in the preceding proof; though this case is rather more simple.

Note that

$$\begin{aligned} & \mathbb{E} \left| \int_t^T A(s)(Y_n(s) - Y(s)) ds \right|_V^2, \\ & \leq T \mathbb{E} \int_0^T |A(s)(Y_n(s) - Y(s))|_V^2 ds \\ & \leq C_6 \mathbb{E} \int_0^T |Y_n(s) - Y(s)|_V^2 ds \\ & \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

where C_6 is some positive constant.

Finally, pass the limit in (3.42) as $n \rightarrow \infty$ to get the equation (3.41). Hence (Y, Z, N) is a solution to (3.41). ■

We now complete the proof of Theorem 3.7.

Proof of Theorem 3.7. The uniqueness proof is similar to that in Lemma 3.8.

Existence : Let $Y_0 \equiv 0$. Define recursively, using Lemma 3.9, the following GBSPDE:

$$\begin{aligned} Y_n(t) = & \xi + \int_t^T (A(s) Y_n(s) + f(s, Z_n(s), Y_{n-1}(s)) + g(s, Y_{n-1}(s))) ds \\ & - \int_t^T Z_n(s) dW(s) - \int_t^T dN_n(s), \end{aligned} \quad (3.51)$$

for $0 \leq t \leq T$ and $n \geq 1$.

The solution to this equation, (Y_n, Z_n, N_n) , lies in $\hat{L}_{\mathcal{F}}^2(0, T; V) \times \Lambda^2(H; \mathcal{P}, W) \times \mathcal{M}^{2,c}(H)$ for each $n \geq 1$, as a result of Lemma 3.9.

The rest of the proof is similar to that in [41]; see also [33].

By using Itô's formula, (F2), (F3) and (F5), one can see easily that

$$\begin{aligned}
& \mathbb{E} |Y_{n+1}(t) - Y_n(t)|_H^2 + \frac{1}{2} \mathbb{E} \int_t^T |Z_{n+1}(s) - Z_n(s)|_{L_2(H)}^2 ds \\
& + \mathbb{E} \int_t^T d \langle N_{n+1} - N_n \rangle_s + \alpha \mathbb{E} \int_t^T |Y_{n+1}(s) - Y_n(s)|_V^2 ds \\
& \leq (\lambda + 2k) \mathbb{E} \int_t^T [|Y_{n+1}(s) - Y_n(s)|_H^2 + \\
& |Y_n(s) - Y_{n-1}(s)|_H^2] ds, \quad 0 \leq t \leq T.
\end{aligned} \tag{3.52}$$

Define

$$u_n(t) := \mathbb{E} \int_t^T |Y_n(s) - Y_{n-1}(s)|_H^2 ds, \quad t \in [0, T], \quad n \geq 1.$$

Then (3.52) implies that

$$-\frac{d}{dt} u_{n+1}(t) - (\lambda + 2k) u_{n+1}(t) \leq (\lambda + 2k) u_n(t),$$

or equivalently

$$-\frac{d}{dt} (u_{n+1}(t) \cdot e^{(\lambda+2k)t}) \leq (\lambda + 2k) e^{(\lambda+2k)t} u_n(t).$$

Thus by integrating both sides of this inequality from t to T , we get

$$u_{n+1}(t) \leq (\lambda + 2k) \int_t^T e^{(\lambda+2k)s} u_n(s) ds.$$

Iterating this inequality yields

$$u_{n+1}(t) \leq [(\lambda + 2k) e^{(\lambda+2k)T}]^n \frac{(T-t)^n}{n!} u_1(0).$$

Hence $\sum_{n=1}^{\infty} u_{n+1}(0)$ is convergent. Therefore by using this result in (3.52), we find that the sequences $\{Y_n\}$, $\{Z_n\}$ and $\{N_n\}$, where $n \geq 1$, are Cauchy in $\hat{L}_{\mathcal{F}}^2(0, T; V)$, $\Lambda^2(H; \mathcal{P}, W)$ and $\mathcal{M}^{2,c}(H)$, respectively. Let Y , Z and N denote their limits respectively.

The strong orthogonality between N and W is showed as in the proof of the previous lemma.

Finally, by using this convergence together with assumptions (F2) and (F5), we can pass the limit in (3.51), as $n \rightarrow \infty$, to get

$$\begin{aligned} Y(t) &= \xi + \int_t^T (A(s) Y(s) + f(s, Y(s), Z(s)) + g(s, Y(s))) ds \\ &\quad - \int_t^T Z(s) dW(s) - \int_t^T dN(s) . \end{aligned}$$

This shows that (Y, Z, N) is a solution to (3.19). ■

We close this section by giving the following example. The reader may find also other examples in this respect in [53, Chapter 4].

Example 3.10 Let $V = \mathbb{H}^1(\mathbb{R}^d)$, $H = L^2(\mathbb{R}^d; \mathbb{R})$ and $V' = \mathbb{H}^{-1}(\mathbb{R}^d)$, where $\mathbb{H}^1(\mathbb{R}^d)$ is the completion of $C_0^\infty(\mathbb{R}^d)$ under the norm :

$$\|u\|_{\mathbb{H}^1(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |u(x)|^2 dx + \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}}$$

and $\mathbb{H}^{-1}(\mathbb{R}^d)$ is the dual space of $\mathbb{H}^1(\mathbb{R}^d)$.

Then (V, H, V') is a rigged Hilbert space; cf. e.g. [53].

Suppose that the $a_{ij}(\omega, t, x)$, $(i, j = 1, \dots, d)$, are bounded real valued processes, defined on $\Omega \times [0, T] \times \mathbb{R}^d$, that are predictable and measurable with respect to the x -variable. Assume moreover that they satisfy the following uniform parabolicity condition: $\exists \delta > 0$ such that

$$-2 \sum_{i,j=1}^d a_{ij}(\omega, t, x) \zeta_i \zeta_j + \delta \sum_{i,j=1}^d \zeta_j^2 \leq 0, \quad (3.53)$$

for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$ and $\zeta_1, \dots, \zeta_d \in \mathbb{R}$.

Consider the following problem :

$$\left\{ \begin{array}{l} -dY(t, x) = \left[\sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(\omega, t, x)) \frac{\partial}{\partial x_j} Y(t, x) \right. \\ \quad \left. + \sum_{j=1}^d b_j(\omega, t, Y(t, x), Z(t, x)) \right] dt \\ \quad - Z(t, x) dW(t) - dN(t, x), \quad 0 \leq t \leq T \\ Y(T, x) = \phi(x) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}), \end{array} \right. \quad (3.54)$$

where $b_j : \Omega \times [0, T] \times \mathbb{R} \times L_2(H; \mathbb{R}) \rightarrow \mathbb{R}$, $j = 1, \dots, d$, satisfies the following two conditions: for each $j = 1, \dots, d$,

- (F1) : b_j is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(L_2(H; \mathbb{R}))/\mathcal{B}(\mathbb{R})$ - measurable and $\mathbb{E} \int_0^T |b_j(t, 0, 0)|_{\mathbb{R}}^2 dt < \infty$;
- (F2) : $\exists k_j > 0$ such that $\forall y, y' \in \mathbb{R}, \forall z, z' \in L_2(H; \mathbb{R})$

$$|b_j(t, y, z) - b_j(t, y', z')|_{\mathbb{R}}^2 \leq k_j (|y - y'|_{\mathbb{R}}^2 + |z - z'|_{L_2(H; \mathbb{R})}^2),$$

uniformly in (t, ω) .

A solution to (3.54) is a triple (Y, Z, N) such that for all $v \in C_0^\infty(\mathbb{R}^d)$ and $t \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} Y(t, x) v(x) dx &= \int_{\mathbb{R}^d} \phi(x) v(x) dx \\ &- \int_t^T \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(\omega, s, x) \frac{\partial}{\partial x_j} Y(s, x) \frac{\partial}{\partial x_i} v(x) dx ds \\ &- \int_t^T \int_{\mathbb{R}^d} \left(\sum_{j=1}^d b_j(\omega, s, Y(s, x), Z(s, x)) \right) v(x) dx ds \\ &- \int_t^T \int_{\mathbb{R}^d} v(x) Z(s, x) dx dW(s) - \int_{\mathbb{R}^d} \int_t^T v(x) dN(s, x) dx. \end{aligned} \quad (3.55)$$

Let us try to relate (3.54) to the GBSPDE (3.19).

Let $A(t, \omega)$ be defined such that

$$[A(t, \omega) u, v] := - \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(\omega, t, x) \frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} v(x) dx ,$$

where $u, v \in V$.

Then from the condition (3.53) we have

$$\begin{aligned} 2 [A(t, \omega) v, v] &\leq - \delta \int_{\mathbb{R}^d} \sum_{i=1}^d \left(\frac{\partial}{\partial x_i} v(x) \right)^2 dx \\ &= \delta \int_{\mathbb{R}^d} |v(x)|_{\mathbb{R}}^2 dx - \delta \left(\int_{\mathbb{R}^d} |v(x)|_{\mathbb{R}}^2 dx + \int_{\mathbb{R}^d} \sum_{i=1}^d \left(\frac{\partial}{\partial x_i} v(x) \right)^2 dx \right) . \end{aligned}$$

Thus A satisfies the condition (F3).

On the other hand, let the mapping $f : \Omega \times [0, T] \times H \times L_2(H) \rightarrow H$ be defined as

$$f(\omega, t, y, z)(x) := \sum_{j=1}^d b_j(\omega, t, y(x), z(x)) ,$$

$(\omega, t, y, z) \in \Omega \times [0, T] \times H \times L_2(H) \rightarrow H$. Then, by using (F1)' and (F2)', it easily seen that this f verifies the conditions (F1) and (F2).

Therefore equation (3.54) can be considered as a GBSPDE of the type (3.19). Consequently, an $\hat{L}_{\mathcal{F}}^2(0, T; H) \times \Lambda^2(H; \mathbb{P}, W) \times \mathcal{M}_{[0, T]}^2(H)$ -valued solution $\{(Y(t, x), Z(t, x), N(t, x)), x \in \mathbb{R}^d\}$ of (3.54) exists uniquely such that $Y(t, \cdot) \in V$ for a.e. (ω, t) and (3.55) holds a.s. for all $t \in [0, T]$.

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